İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

INVARIANT SUBSPACE THEOREMS FOR FAMILIES OF OPERATORS ON BANACH SPACES AND BANACH LATTICES

Ph.D. Thesis by

R. Tunç MISIRLIOĞLU, M.Sc.

Department: **Mathematics**

Programme: **Mathematical Engineering**

JUNE 2006

İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

INVARIANT SUBSPACE THEOREMS FOR FAMILIES OF OPERATORS ON BANACH SPACES AND BANACH LATTICES

Ph.D. Thesis by
R. Tunç MISIRLIOĞLU, M.Sc.
(509992001)

Date of submission: 24 February 2006
Date of defence examination: 20 June 2006

Supervisor (Chairman): **Asist.Prof.Dr. Recep KOKRMAZ**

Co-Supervisor: **Prof.Dr. Şafak ALPAY (METU)**

Members of the Examining Committee **Prof.Dr. Ulviye BAŞER (ITU)**

Prof.Dr. Faruk GÜNGÖR (ITU)

Prof.Dr. Betül TANBAY (Boğaziçi Univ.)

Assoc.Prof.Dr. Zafer ERCAN (METU)

Assoc.Prof.Dr. Mahir HASANOV (ITU)

JUNE 2006

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

BANACH UZAYLARI VE BANACH ÖRGÜLERİ ÜZERİNDE TANIMLI OPERATÖR AİLELERİ İÇİN DEĞİŞMEZ ALTUZAY TEOREMLERİ

DOKTORA TEZİ

R. Tunç MISIRLIOĞLU
(509992001)

Tezin Enstitüye Verildiği Tarih: 24 Şubat 2006
Tezin Savunulduğu Tarih: 20 Haziran 2006

Tez Danışmanı:	**Yard.Doç.Dr. Recep KORKMAZ**
Ortak Danışman:	**Prof.Dr. Şafak ALPAY (ODTÜ)**
Diğer Jüri Üyeleri	**Prof.Dr. Ulviye BAŞER (İTÜ)**
	Prof.Dr. Faruk GÜNGÖR (İTÜ)
	Prof.Dr. Betül TANBAY (Boğaziçi Ü.)
	Doç.Dr. Zafer ERCAN (ODTÜ)
	Doç.Dr. Mahir HASANOV (İTÜ)

HAZİRAN 2006

PREFACE

I would like to thank Prof. Dr. Şafak Alpay for introducing me to Invariant Subspace Problem and for his numerous suggestions and improvements. His support and his excellent mathematical judgment have helped me to express myself mathematically.

I am very grateful to my wife Selda and our daughter Ece for their patience, love and support.

June 2006 R. Tunç Mısırlıoğlu

CONTENTS

INVARIANT SUBSPACE THEOREMS
FOR FAMILIES OF OPERATORS ON
BANACH SPACES AND BANACH LATTICES

SUMMARY

In this thesis, the invariant subspace problem is studied for certain families of linear bounded operators on Banach spaces. We also consider families of positive operators on Banach lattices.

This thesis is divided into four chapters.

In the first chapter, we give a brief history of the invariant subspace problem. Then the relation of the work in the thesis with the existing works on this subject is established.

The notion of a compact-friendly operator was introduced in 1994 by Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw. They proved that every non-zero locally quasinilpotent compact-friendly operator on a Dedekind complete Banach lattice has a non-trivial closed invariant ideal. In Chapter 2, we show that Dedekind completeness is not necessary and that the Banach lattice having separating orthomorphisms is sufficient. We also generalize a theorem due to R. Drnovsek in 2001 by using the concept of compact-friendliness.

In Chapter 3, we give an invariant subspace result for a semigroup of positive operators on a Banach space with a (Schauder) basis. This is a generalization of a theorem of Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw in 1995. This theorem states that for a positive continuous operator defined on a Banach space with a basis if it commutes with a non-zero positive locally quasinilpotent operator, then it has a non-trivial closed invariant subspace. By using the notion of weakly quasinilpotence introduced in 2004 by Z. Ercan and S. Önal, we then generalize our result to topological vector spaces with Markushevich basis.

In Chapter 4, collectively compact sets of linear bounded operators on infinite dimensional Banach spaces are studied in connection with the invariant subspace problem. We give some invariant subspace results with respect to the joint spectral radius and its local version for these sets. It is also shown, in a special case, that any collectively compact set of operators satisfies the Berger-Wang formula.

BANACH UZAYLARI VE BANACH ÖRGÜLERİ ÜZERİNDE TANIMLI OPERATÖR AİLELERİ İÇİN DEĞİŞMEZ ALTUZAY TEOREMLERİ

ÖZET

Bu tez çalışmasında, Banach uzayları üzerinde tanımlı doğrusal sınırlı operatörlerin oluşturduğu bazı aileler ile Banach örgüleri üzerinde tanımlı pozitif operatörlerin oluşturduğu bazı aileler için değişmez altuzay problemi incelenmiştir.

Bu çalışma dört bölüme ayrılmıştır.

Birinci bölümde, değişmez altuzay probleminin kısa bir tarihi verilmiştir. Daha sonra, bu tez çalışmasında gözönüne alınan konuların, aynı konularda var olan çalışmalar ile ilişkisi irdelenmiştir.

Kompakt-yakın operatör kavramı ilk olarak 1994 yılında Y.A. Abramovich, C.D. Aliprantis ve O. Burkinshaw tarafından tanımlanmış olup, bir Dedekind tam Banach örgüsü üzerinde tanımlı yerel yarınilpotent ve kompakt-yakın olan her sıfırdan farklı operatörün, aşikâr olmayan kapalı değişmez bir ideale sahip olduğu gösterilmiştir. Bu tez çalışmasının ikinci bölümünde, Banach örgüsünün Dedekind tam olması yerine, bundan daha geniş bir özellik olan Banach örgüsünün ortomorfizmaları ayırma özelliğine sahip olmasının yeterli olduğu gösterilmektedir. Ayrıca, 2001 yılında R. Drnovsek'in bir teoremi, kompakt-yakınlık kavramı kullanılarak genelleştirilmektedir.

Üçüncü bölümde, Schauder tabanına sahip bir Banach uzayı üzerinde tanımlı pozitif operatörlerden oluşan yarıgruplar için bir değişmez altuzay teoremi verilmektedir. Bu teorem, Y.A. Abramovich, C.D. Aliprantis ve O. Burkinshaw'ın 1995 yılında tek bir operatör için gösterdikleri, Schauder tabanına sahip bir Banach uzayı üzerinde tanımlı pozitif bir operatörün, bu operatör ile değişmeli olan sıfırdan farklı bir pozitif operatörün yerel yarınilpotent olması durumunda, aşikâr olmayan kapalı değişmez bir altuzaya sahip olduğunu belirten teoreminin bir genelleştirilmesidir. Ayrıca bu teorem, Z.Ercan ve S.Önal tarafından 2004 yılında tanımlanan zayıf yarınilpotentlik kavramı kullanılarak, Markushevich tabanına sahip topolojik vektör uzaylarına genişletilmektedir.

Dördüncü bölümde ise, Banach uzayları üzerinde tanımlı doğrusal sınırlı operatörlerden oluşan birlikte kompakt kümeler, değişmez altuzay problemi ile bağlantılı olarak ele alınmaktadır. Birlikte kompakt kümeler için, ortak spektral yarıçap ve bunun yerel versiyonuna göre, bazı değişmez altuzay teoremleri verilmektedir. Ayrıca, birlikte kompakt kümelerin, özel bir durumda, Berger-Wang formülünü gerçeklediği gösterilmektedir.

1. INTRODUCTION

An invariant subspace problem is the simple question: ”Does every bounded linear operator T on a separable Hilbert space H over complex numbers $\mathbb{C}$ have a non-trivial invariant subspace?” Here non-trivial subspace means a closed subspace of H different from $\{0\}$ and H. Invariant means that the operator T maps it to itself; in other words, a subspace V of H is T-invariant if $T(V) \subseteq V$.

The problem is easy to state, however, it is still open. The answer is ’no’ in general for (separable) complex Banach spaces. For certain classes of bounded linear operators on complex Hilbert or Banach spaces, the problem has an affirmative answer.

In 1954, Aronszajn and Smith [10] proved that every compact operator has a non-trivial invariant subspace.

In 1973, V. Lomonosov obtained a more general result [23]: If a non-scalar (i.e., not a multiple of the identity operator) bounded operator T on a Banach space X commutes with a non-zero compact operator, then T has a non-trivial hyperinvariant subspace. Here a subspace V of X is called T-hyperinvariant if V is invariant under every continuous operator that commutes with T. In 1980, however, Hadvin-Nordgren-Radjavi-Rosenthal gave an example of an operator that does not commute with any non-zero compact operator [20]. Lomonosov’s theorem highlighted another, stronger, form of the ’invariant subspace problem’: ”Does every bounded linear operator on a Hilbert space have a non-trivial hyperinvariant subspace?”

In 1976, Per Enflo showed the existence of a Banach space and a bounded linear operator on it without any non-trivial invariant subspace by giving an example [17]. Due to this counterexample, the invariant subspace problem for operators on Banach spaces has been confined to the search for various classes of operators for which one can guarantee the existence of an invariant subspace.

With respect to invariant subspaces of individual operators, there are several existence theorems in addition to those related to Lomonosov's. Scott Brown, in 1978, proved that subnormal operators (operators that are restrictions of normal operators to invariant subspaces) on Hilbert spaces have non-trivial closed invariant subspaces [13]. Recall that a linear bounded operator T on a Hilbert space is said to be normal if it commutes with its adjoint, i.e., $TT^* = T^*T$.

In 1985, C.J. Read also constructed a counterexample [27]. Building on his work, Read also gave an example of a quasinilpotent bounded operator on a Banach space without a non-trivial invariant subspace [28].

The first appearance of the joint spectral radius of operator families in the invariant subspace theory was in 1984, V.S. Shulman proved that every algebra of Volterra (i.e., compact quasinilpotent) operators on a Banach space has a common invariant subspace [33]. In 1999, Yu.V. Turovskii answered the long standing open question: Every multiplicative semigroup of Volterra operators on a Banach space has a common invariant subspace [35].

For positive operators on Banach lattices, the invariant subspace problem is still open: "Does every positive operator on a Banach lattice of dimension greater than two has a non-trivial closed invariant subspace?"

In 1986, Ben de Pagter proved that every positive quasinilpotent compact operator on a Banach lattice of dimension greater than one has a non-trivial closed ideal [25]. Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw in [2] extended the result of B. de Pagter: If a locally quasinilpotent positive operator S dominates a non-zero compact operator, then every positive operator that commutes with S has a non-trivial closed invariant ideal. Generalizing the result of Ben de Pagter and using the result in [2], R. Drnovsek proved in [16] that a semigroup of positive Volterra operators on a Banach lattice has a non-trivial hyperinvariant ideal. It was also proved in [16] that a locally finitely quasinilpotent family of positive operators on a Banach lattice whose commutant contains an operator that dominates a non-zero compact operator has a non-trivial closed invariant ideal. Furthermore, V.S. Shulman and Yu.V. Turovskii generalized these results: For a locally finitely quasinilpotent family of operators on a Banach space, if a closed subalgebra generated by this family and by its commutant contains a non-zero compact operator, then the family has a non-trivial hyperinvariant subspace [34].

In this thesis, we study on the invariant subspace problem for certain families of linear bounded operators on Banach spaces and families of positive operators on Banach lattices.

In Chapter 2, we show that every non-zero locally quasinilpotent compact-friendly operator on a Banach lattice with separating orthomorphisms has a non-trivial closed invariant ideal. Also, we generalize the second result of R. Drnovsek expressed above by using the notion of compact-friendliness.

In Chapter 3, we prove that a locally finitely quasinilpotent multiplicative semigroup of positive continuous operators defined on a Banach space with a Schauder basis has a non-trivial closed invariant subspace. We then generalize our result to topological vector spaces with Markushevich basis by using the notion of weakly quasinilpotence introduced by Z. Ercan and S. Önal in [18].

In Chapter 4, collectively compact sets of linear bounded operators on infinite dimensional Banach spaces are considered in connection with the invariant subspace problem. We give some invariant subspace results for these sets with respect to the joint spectral radius and its local version. It is also shown that any collectively compact set M in $\text{alg}\Gamma$ satisfies the Berger-Wang formula, where Γ is a complete chain of subspaces of X and $\text{alg}\Gamma$ denotes the set of operators that leave all the subspaces in Γ invariant.

2. FAMILIES OF POSITIVE OPERATORS ON A BANACH LATTICE WITH SEPARATING ORTHOMORPHISMS

A subset C of a (real or complex) vector space V is said to be a cone whenever $C + C \subseteq C$, $\alpha C \subseteq C$ for each real $\alpha \geqslant 0$, and $C \cap (-C) = \{0\}$. Every cone C induces a partial order $\geqslant$ on V as follows: For $x, y \in V$ we say that

$$x \leqslant y \Leftrightarrow y - x \in C.$$

So, $V^+ = C = \{x \in V : x \geqslant 0\}$. The elements of C are referred to as positive vectors. A (partially) ordered vector space (V, C) is a vector space V equipped with a cone C. An ordered vector space is said to be Archimedean if $nx \leqslant y$ for all n implies $x \leqslant 0$.

A bounded linear operator $T : V \to V$ on an ordered vector space (V, C) is said to be positive if $T(C) \subseteq C$. In other words, $Tx \geqslant 0$ for each $x \geqslant 0$. In this case, T is increasing; i.e., $x \leqslant y \Rightarrow Tx \leqslant Ty$.

A lattice ordered vector space E is called a Riesz space (or a vector lattice). That is, an ordered vector space E is a Riesz space if every pair of vectors has a supremum and an infimum; as a notation,

$$x \vee y := \sup\{x, y\} \quad \text{and} \quad x \wedge y := \inf\{x, y\}.$$

For an element x in a Riesz space, its positive part, its negative part, and its absolute value are defined by

$$x^+ = x \vee 0,\ x^- = (-x) \vee 0,\ |x| = x \vee (-x),$$

respectively. We have the following two important identities:

$$x = x^+ - x^- \quad \text{and} \quad |x| = x^+ + x^-.$$

The functions $(x, y) \mapsto x \vee y$, $(x, y) \mapsto x \wedge y$, $x \mapsto x^+$, $x \mapsto x^-$, and $x \mapsto |x|$ are referred to collectively as the lattice operations of a Riesz space.

A Riesz space is said to be Dedekind complete whenever every non-empty subset of the space that is bounded from above has a supremum. Similarly, a Riesz space is said to be σ-Dedekind complete whenever every non-empty countable subset of the space that is bounded from above has a supremum.

Let E be a Riesz space. A subset A of E is said to be solid if $|x| \leqslant |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace of E is called an ideal. The ideal generated by a non-empty subset A of E is the smallest (with respect to inclusion) ideal containing A; it coincides with the intersection of all ideals that contain A and it is given by

$$E_A = \{x \in E : \exists x_1, ...x_n \in A \quad \text{and} \quad \lambda_1, ...\lambda_n \in \mathbb{R}^+ \text{ with } |\mathrm{x}| \leqslant \sum_{\mathrm{i}=1}^{\mathrm{n}} \lambda_{\mathrm{i}} |\mathrm{x}_{\mathrm{i}}|\}.$$

If $A = \{x\}$, then $E_x = E_{\{x\}}$ is called the principal ideal generated by the vector x. Note that

$$E_x = \{y \in E : \exists \lambda \geqslant 0 \quad \text{s.t.} \quad |y| \leqslant \lambda |x|\}.$$

A vector $e > 0$ is called an order unit if $E_e = E$; i.e., if for each $x \in E$ there exists some $\lambda \geqslant 0$ such that $|x| \leqslant \lambda e$.

A seminorm p on a Riesz space is said to be a lattice seminorm whenever $|x| \leqslant |y|$ implies $p(x) \leqslant p(y)$. A normed Riesz space is a Riesz space equipped with a lattice norm. A normed Riesz space which is also norm complete (i.e., a Banach space) is called a Banach lattice.

An operator $T : E \to E$ on a real or complex Riesz space is called central if it is dominated by a multiple of the identity operator. That is, T is a central operator if and only if there exists some scalar $\lambda > 0$ such that $|Tx| \leqslant \lambda |x|$ holds for all $x \in E$. The collection of all central operators is denoted by $Z(E)$ and is referred to as the center of the Banach lattice E. The central operators are special examples of orthomorphisms. An order bounded operator $T : E \to E$ on a Riesz space is said to be an orthomorphism if $|x| \wedge |y| = 0$ implies $|x| \wedge |Ty| = 0$. Orth (E) will denote the orthomorphisms of E.

Definition 2.1: A Riesz space E is said to have separating orthomorphisms if the following holds: If $x \wedge y = 0$, then there exists $\pi \in \mathrm{Orth}(E)$ such that $\pi(x) = x$ and $\pi(y) = 0$. Or equivalently, if for all $x \in E$ there exists $\pi \in \mathrm{Orth}(E)$ such that

$\pi(x^+) = x^+$ and $\pi(x^-) = 0$. We note that if $x \wedge y = 0$ and $\pi \in \mathrm{Orth}(E)$ satisfies $\pi(x) = x$ and $\pi(y) = 0$, then the orthomorphism $\pi_1 = |\pi| \wedge I$ satisfies $\pi_1(x) = x$ and $\pi_1(y) = 0$. Hence, we may assume that $0 \leqslant \pi \leqslant I$ in the definition.

An order ideal I in E has the extension property if every π_0 in $Z(I)$ has an extension $\pi \in Z(E)$. The following Theorem is due to de Pagter [24].

Theorem 2.2: Consider the following three conditions in an Archimedean Riesz space E.

a) If $0 \leqslant u \leqslant v$ in E, then $u = \pi(v)$ for some $0 \leqslant \pi$ in Orth (E).

b) Every principal order ideal in E has the extension property.

c) E has separating orthomorphisms.

Then $a \Rightarrow b \Rightarrow c$. Moreover, if E is, in addition, uniformly complete, then a, b, and c are equivalent.

Every σ-Dedekind complete Riesz space has both the extension property and separating orthomorphisms. However, every order ideal in a uniformly complete Riesz space E has the extension property if and only if E is Dedekind complete. This is due to Wickstead [36].

Let X be an infinite dimensional complex Banach space and $B(X)$ denote the space of all linear bounded operators on X.

The spectrum $\sigma(T)$ of an operator $T \in B(X)$ is the set of all complex numbers λ such that the operator $\lambda I - T$ is not invertible on X. The complement of the spectrum is called the resolvent set of T.

The spectral radius $\rho(T)$ of an arbitrary operator T in $B(X)$ is the smallest non-negative real number r for which the closed disk $\{\lambda \in C : |\lambda| \leqslant r\}$ contains the spectrum $\sigma(T)$. The following formula is known as Gelfand formula

$$\rho(T) = \lim_{n\to\infty} \| T^n \|^{1/n} .$$

An operator $T \in B(X)$ is said to be quasinilpotent if $\rho(T) = 0$ and locally quasinilpotent at x_0 if

$$\lim_{n\to\infty} \| T^n x_0 \|^{1/n} = 0.$$

For a norm bounded subset B of a Banach space X and for a norm bounded family $\mathcal{C}$ of linear bounded operators in $B(X)$, we let

$$\|B\| := \sup\{\|x\| : x \in B\} \quad \text{and} \quad \|\mathcal{C}\| := \sup\{\|T\| : T \in \mathcal{C}\}.$$

Denote $\mathcal{C}B := \{Tx : T \in \mathcal{C}, x \in B\}$. If $B = \{x\}$, we will then write $\mathcal{C}x$ instead of $\mathcal{C}\{x\}$. The powers $\mathcal{C}^n$ are defined inductively by $\mathcal{C}^1 = \mathcal{C}$, $\mathcal{C}^n = \mathcal{C}\mathcal{C}^{n-1}$ for $n \geqslant 2$.

$$\mathcal{C}^n = \{C_1C_2...C_n : \ C_1, C_2, ..., C_n \in \mathcal{C}\}$$

and the norm of it is the supremum of the norms of its elements

$$\|\mathcal{C}^n\| = \sup\{\|S\| : \ S \in \mathcal{C}^n\}.$$

The commutant of a family $\mathcal{C}$ will be denoted by $\mathcal{C}'$. A subspace V of X is said to be $\mathcal{C}$-invariant if V is T-invariant for each $T \in \mathcal{C}$.

A family $\mathcal{C}$ of operators in $B(X)$ is said to be (locally) quasinilpotent at a point $x \in X$ if $\lim_{n\to\infty} \| \mathcal{C}^n x \|^{1/n} = 0$ and finitely quasinilpotent at a point $x \in X$ if every finite subcollection of $\mathcal{C}$ is (locally) quasinilpotent at x.

A family $\mathcal{S}$ of linear operators on a Banach space X is said to be multiplicative semigroup if for each pair $S, T \in \mathcal{S}$, $ST \in \mathcal{S}$. A subset $\mathcal{S}_0$ of a multiplicative semigroup $\mathcal{S}$ is said to be a (two-sided algebraic) semigroup ideal in $\mathcal{S}$ if for each $T \in \mathcal{S}_0$ and for each $S \in \mathcal{S}$ the operators TS and ST belong to $\mathcal{S}_0$.

For the rest of this section we shall assume that $\mathcal{C}$ denotes a non-empty collection of positive operators on a Banach lattice E. For a positive operator $T : E \to E$ on a Banach lattice E, we denote by $\mathrm{RC}(T)$ the collection of all positive operators $S : E \to E$ such that $ST \geqslant TS$. In accordance with this notation we also let

$$\mathrm{RC}(\mathcal{C}) = \{S : S \geqslant 0 \quad \text{and} \quad ST \geqslant TS \quad \text{for each} \quad T \in \mathcal{C}\}.$$

For a collection $\mathcal{C}$ of positive operators on E, the smallest semigroup that contains $\mathcal{C}$ will be denoted by $\mathrm{SG}(\mathcal{C})$. Given a family $\mathcal{C}$, the collection $\mathcal{D}_{\mathcal{C}}$ of positive operators is defined as follows:

$$\mathcal{D}_{\mathcal{C}} = \{D : D \geqslant 0, \exists\{T_1, \cdots, T_k\} \subseteq \mathrm{RC}(\mathcal{C}), \{S_1, \cdots, S_k\} \subseteq \mathrm{SG}(\mathcal{C}) \quad \text{s.t.} \quad D \leqslant \sum_{i=1}^{k} T_iS_i\}.$$

We refer the reader to [1] for properties of $\mathrm{SG}(\mathcal{C})$ and $\mathcal{D}_{\mathcal{C}}$.

Definition 2.3: A positive operator $B : E \to E$ on a Banach lattice is called compact-friendly if there exists a positive operator in the commutant of B that dominates a non-zero operator that is dominated by a compact positive operator. That is, there exist non-zero operators $R, K, C : E \to E$ with R positive, K positive compact, and satisfying $RB = BR$, $|Cx| \leqslant R|x|$, and $|Cx| \leqslant K|x|$ for each $x \in E$.

The following was given in [1] as Theorem 10.57.

Theorem 2.4: If a non-zero compact-friendly operator $B : E \to E$ on a Dedekind complete Banach lattice is quasinilpotent at some $x_0 > 0$, then there exists a non-trivial closed ideal that is invariant under $\mathrm{RC}(B)$.

The next result is a generalization of the preceding Theorem. We show that Dedekind completeness is not needed and that E having separating orthomorphisms is sufficient. The proof is a modification of the proof of Theorem 10.57 in [1].

Proposition 2.5: Let E be a Banach lattice with separating orthomorphisms. If a non-zero compact-friendly operator $B : E \to E$ is quasinilpotent at some $x_0 > 0$, then there exists a non-trivial closed ideal that is invariant under $\mathrm{RC}(B)$.

Proof. For each $0 < x$, we denote by J_x the ideal generated by the orbit $\mathrm{RC}(B)x$,

$$J_x = \{y \in E : \quad |y| \leqslant Ax \quad \text{for some} \quad A \in \mathrm{RC}(B)\}.$$

As $x \in J_x$, J_x is a non-zero ideal and J_x is $\mathrm{RC}(B)$-invariant. Therefore, if for some vector $x > 0$ the ideal J_x is not norm dense in E, then $\overline{J}_x$ is a non-trivial closed $\mathrm{RC}(B)$-invariant ideal.

Consequently, we assume from now on that $\overline{J}_x = E$ for each $x > 0$.

B is compact-friendly, hence there exist non-zero operators $R, K, C : E \to E$ with R positive, K positive compact, and satisfying $RB = BR$, $|Cx| \leqslant R|x|$, and $|Cx| \leqslant K|x|$ for each $x \in E$.

Since $C \neq 0$, there exists some $x_1 > 0$ such that one of the vectors $(Cx_1)^+$ or $(Cx_1)^-$ is non-zero. Suppose that $(Cx_1)^+ \neq 0$. Then there exists an orthomorphism M, $0 \leqslant M \leqslant I$, of E with $MCx_1 > 0$. Let $x_2 = MCx_1$ and

$\pi_1 = MC$. We note that the operator π_1 is dominated by the compact operator K and the positive operator R.

Since J_{x_2} is dense in E and C is non-zero, there exists $y \in J_{x_2}$ and $A_1 \in \mathrm{RC}(B)$ such that $0 < y < A_1 x_2$ and $Cy \neq 0$. E has separating orthomorphisms, thus there exists an operator M_1, $0 \leqslant M_1 \leqslant I$, in $Z(E)$, with $y = M_1 A_1 x_2$. Suppose that $(CM_1A_1x_2)^+ \neq 0$. We choose M_2 in $Z(E)$ with $0 \leqslant M_2 \leqslant I$, such that $M_2CM_1A_1x_2 = (CM_1A_1x_2)^+ > 0$. Letting $x_3 = M_2CM_1A_1x_2$ and $\pi_2 = M_2CM_1A_1$, we note that the operator π_2 is dominated by the compact operator KA_1 and the positive operator RA_1.

We now repeat the preceding arguments with the vector x_2 replaced by x_3. There exists some z in J_{x_3} and an operator A_2 in $\mathrm{RC}(B)$ such that $0 < z \leqslant A_2x_3$ and $Cz \neq 0$. We choose M_3 in $Z(E)$, $0 \leqslant M_3 \leqslant I$, with $z = M_3A_2x_3$. Suppose $(CM_3A_2x_3)^+$ is not zero. We choose M_4 in $Z(E)$, $0 \leqslant M_4 \leqslant I$ with $M_4CM_3A_2x_3 = (CM_3A_2x_3)^+ > 0$. Now let $\pi_3 = M_4CM_3A_2$.

Consider the operator $\pi_3\pi_2\pi_1$. It is non-zero as $\pi_3\pi_2\pi_1x_1 = \pi_3x_3 \neq 0$. Each operator π_i is dominated by a compact positive operator and therefore $\pi_3\pi_2\pi_1$ is compact by Theorem 2.34 in [1]. Also for each $x \in E$, we have $|\pi_3\pi_2\pi_1x| \leqslant RA_2RA_1R|x|$.

If $S = RA_2RA_1R$, then $S \in \mathrm{RC}(B)$ since $\mathrm{RC}(B)$ is a semigroup. Consider $\mathcal{C} = \{B\}$. The family $\mathcal{D}_B$ is finitely quasinilpotent at x_0 by Lemma 10.43 in [1]. $\mathrm{RC}(B)$ is contained in the family $\mathcal{D}_B$. Therefore $\mathrm{RC}(B)$ is finitely quasinilpotent at x_0 and contains the operator S that dominates a non-zero compact operator $\pi_3\pi_2\pi_1$. That $\mathrm{RC}(B)$ has a non-trivial closed invariant ideal follows from Theorem 10.44 in [1]. □

A Theorem due to Drnovsek [1, Theorem 10.50] says that if $\mathcal{C}$ is a family of positive operators on a Banach lattice which is finitely quasinilpotent at a non-zero positive vector and its commutant $\mathcal{C}'$ contains an operator that dominates a non-zero compact operator, then the families $\mathcal{C}$ and $\mathrm{RC}(\mathcal{C})$ have a common non-trivial closed invariant ideal.

The next result generalizes the preceding and shows that $\mathcal{D}_{\mathcal{C}}$ and $\mathrm{RC}(\mathcal{D}_{\mathcal{C}})$ have a common non-trivial closed ideal.

Proposition 2.6: Let $\mathcal{C}$ be a non-zero collection of positive operators on a Banach lattice E with separating orthomorphisms. Suppose $\mathcal{C}$ is finitely quasinilpotent at some $x_0 > 0$ and the commutant $\mathcal{C}'$ of $\mathcal{C}$ contains a positive operator T_0 which dominates an operator L which is dominated by a compact positive operator K (i.e., there exist L and K (positive, compact) with $|Lx| \leqslant T_0|x|$ and $|Lx| \leqslant K|x|$ for all $x \in E$). Then $\mathcal{C}$ and $\mathrm{RC}(\mathcal{C})$ have a common non-trivial closed invariant ideal.

Proof: We know that the family $\mathcal{D}_{\mathcal{C}}$ is finitely quasinilpotent at x_0 by Lemma 10.43 in [1]. For $x > 0$, we denote by $[\mathcal{D}_{\mathcal{C}}x]$ the ideal generated by the orbit $\mathcal{D}_{\mathcal{C}}x$ of the family $\mathcal{D}_{\mathcal{C}}$, i.e., $[\mathcal{D}_{\mathcal{C}}x] = \{y \in E : |y| \leqslant Dx \text{ for some } D \in \mathcal{D}_{\mathcal{C}}\}$. Since $I \in \mathcal{D}_{\mathcal{C}}$, $x \in [\mathcal{D}_{\mathcal{C}}x]$ and $[\mathcal{D}_{\mathcal{C}}x] \neq \{0\}$ if $x > 0$. It is easy to show that $[\mathcal{D}_{\mathcal{C}}x]$ is invariant under $\mathcal{D}_{\mathcal{C}}$. Therefore, we may assume that $\overline{[\mathcal{D}_{\mathcal{C}}x]} = E$ for $x > 0$.

As $L \neq 0$, there exists x_1 with $Lx_1 \neq 0$. Thus, either $(Lx_1)^+$ or $(Lx_1)^-$ is non-zero. Suppose $(Lx_1)^+ \neq 0$. Since E has separating orthomorphisms, there exists an orthomorphism V, $0 \leqslant V \leqslant I$, such that $VLx_1 = (Lx_1)^+$.

Let $x_2 = VLx_1$ and $M_1 = VL$. Observe that the operator M_1 is dominated by the compact operator K and the operator T_0. Recall that $\overline{[\mathcal{D}_{\mathcal{C}}x_2]} = E$. Hence, there exists y with $0 < y \leqslant D_1\, x_2$ for some $D_1 \in \mathcal{D}_{\mathcal{C}}$ such that $Ly \neq 0$. Since E has separating orthomorphisms, there exists U_1, $0 \leqslant U_1 \leqslant I$, with $y = U_1D_1x_2$. $Ly \neq 0$ therefore there exists an orthomorphism V_1, $0 \leqslant V_1 \leqslant I$, with $V_1Ly > 0$. That is, $V_1LU_1D_1x_2 > 0$. Let $x_3 = V_1LU_1D_1x_2$ and $M_2 = V_1LU_1D_1$. As $|M_2z| = |V_1LU_1D_1z| \leqslant |LU_1D_1z| \leqslant K|U_1D_1z| \leqslant KD_1|z|$ for each $z \in E$, we see that the operator M_2 is dominated by the compact operator KD_1. M_2 is also dominated by the positive operator T_0D_1.

As $\overline{[\mathcal{D}_{\mathcal{C}}x_3]} = E$, we must have $Lz \neq 0$ for some z with $0 < z < D_2x_3$, where $D_2 \in \mathcal{D}_{\mathcal{C}}$. Then $z = U_2D_2x_3$ for some $0 \leqslant U_2 \leqslant I$ and, as $Lz \neq 0$, we have $V_2LU_2D_2x_3 > 0$ for some V_2 with $0 \leqslant V_2 \leqslant I$. Let $M_3 = V_2LU_2D_2$. Then M_3 is dominated by the compact operator KD_2 and also by the positive operator T_0D_2. We have $T_0D_2T_0D_1T_0 \in \mathcal{D}_{\mathcal{C}}$ since $\mathcal{D}_{\mathcal{C}}$ is a multiplicative semigroup and $T_0 \in \mathcal{C}'$.

On the other hand, $|M_3M_2M_1x| \leqslant T_0D_2T_0D_1T_0|x|$ for each $x \in E$. Thus $M_3M_2M_1$ is compact by Theorem 2.34 in [1]. Hence $\mathcal{D}_{\mathcal{C}}$ contains an operator $T_0D_2T_0D_1T_0$ which dominates the compact operator $M_3M_2M_1 \neq 0$. Hence $\mathcal{D}_{\mathcal{C}}$ has a non-trivial closed invariant ideal by Theorem 10.44 in [1]. Since $\mathcal{C}$ and $\mathrm{RC}(\mathcal{C})$ are contained in $\mathcal{D}_{\mathcal{C}}$, it follows that $\mathcal{C}$ and $\mathrm{RC}(\mathcal{C})$ have a common non-trivial closed invariant ideal. $\square$

3. FAMILIES OF POSITIVE OPERATORS ON A BANACH SPACE WITH A (MARKUSHEVICH) BASIS

A sequence $\{x_n\}$ of a Banach space X is called a Schauder basis if for each $x \in X$ there exists a unique sequence of scalars $\{\alpha_n\}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$, where the series is convergent in norm. Associated with the basis the standard sequence of coefficient functionals f_n (n=1,2,...) is defined by

$$f_n(x) = \alpha_n \quad \text{for} \quad x = \sum_{i=1}^{\infty} \alpha_i x_i \in X.$$

Each f_n is linear functional on X and $f_n \in X'$, where X' is the dual of X. Also $f_n(x_m) = \delta_{nm}$.

Every basis $\{x_n\}$ gives rise to a closed cone C defined by

$$C = \{x = \sum_{n=1}^{\infty} \alpha_n x_n : \alpha_n \geqslant 0, \forall n = 1, 2, ...\}.$$

The cone C will be referred to as the cone generated by the basis $\{x_n\}$.

A bounded linear operator $T : X \to X$ on a Banach space X with a basis $\{x_n\}$ is said to be positive (with respect to this basis) if $T(C) \subseteq C$, where C is the cone generated by $\{x_n\}$.

Let X be a Hausdorff topological vector space with dual X'. A sequence (x_n, f_n) in $X \times X'$ is called a Markushevich basis if the span of (x_n) is dense in X, $f_n(x_n) = 1$ and $f_n(x_m) = 0$ for $n \neq m$, and $\{f_n\}$ separates the points of X. Clearly, a Schauder basis in a Banach space is also a Markushevich basis but the converse is not true in general.

A Hausdorff topological vector space with a Markushevich basis can be partially ordered by the relation $x \leqslant y$ if and only if $f_n(x) \leqslant f_n(y)$ for all n.

For a detailed analysis about Schauder and Markushevich bases we refer the reader to [32].

The following theorem was proved in [3], see also [1, Theorem 10.66].

Theorem 3.1: Let $T : X \to X$ be a positive continuous operator defined on a Banach space X with a basis. If T commutes with a non-zero positive operator that is quasinilpotent at a non-zero positive vector, then T has a non-trivial closed invariant subspace.

Now, we give an invariant subspace result for a semigroup of positive operators on a Banach space with a Schauder basis. The proof goes along the same lines as in the proof of Theorem 3.1.

Proposition 3.2: Let $\mathcal{J}$ be a multiplicative semigroup of positive continuous operators defined on a Banach space X. If $\{x_n\}$ is a Schauder basis for X and $\mathcal{J}$ is finitely quasinilpotent at some $x_0 > 0$, then $\mathcal{J}$ has a non-trivial closed invariant subspace.

Proof: Let $\{f_n\}$ be the sequence of coefficient functionals associated with the basis $\{x_n\}$. If $Tx_0 = 0$ for each $T \in \mathcal{J}$, then $\bigcap_{T \in \mathcal{J}} T^{-1}(0)$ is a non-trivial closed subspace that is invariant under $\mathcal{J}$. Thus, we can assume that $Tx_0 \neq 0$ for some $T \in \mathcal{J}$. Hence $Tx_k \neq 0$ for some k. Without loss of generality, we can assume $0 \leqslant x_k \leqslant x_0$.

Let $P : X \to X$ be the continuous projection onto the subspace spanned by x_k defined by $Px = f_k(x)x_k$. Then $0 \leqslant Px \leqslant x$ for each $0 \leqslant x \in X$. We claim that

$$PSTx_k = 0$$

for each $S \in \mathcal{J}$. To prove this, let $PSTx_k = \alpha x_k$ for some non-negative scalar $\alpha \geqslant 0$. Since all the operators involved are positive, we have

$$0 \leqslant \alpha^n x_k \leqslant (PST)^n x_k \leqslant (ST)^n x_k \leqslant (ST)^n x_0.$$

Using the positivity of the functional f_k, we have

$$0 \leqslant \alpha^n = f_k(\alpha^n x_k) \leqslant f_k((ST)^n x_0).$$

Consequently,

$$0 \leqslant \alpha^n \leqslant ||f_k|| \;\; ||(ST)^n x_0||.$$

Since $ST \in \mathcal{J}$ and $\mathcal{J}$ is finitely quasinilpotent at x_0, from

$$0 \leqslant \alpha \leqslant ||f_k||^{\frac{1}{n}} ||(ST)^n x_0||^{\frac{1}{n}}$$

and $\lim_{n\to\infty} ||(ST)^n x_0||^{\frac{1}{n}} = 0$, we have $\alpha = 0$.

Finally, we consider the subspace Y generated by $\{STx_k : \quad S \in \mathcal{J}\}$. We have $Y \neq \{0\}$ since $Tx_k \neq 0$. Since $\mathcal{J}$ is a multiplicative semigroup, Y is invariant under $\mathcal{J}$.

Thus, it remains to show $\overline{Y} \neq X$. For each $y \in Y$, we have

$$f_k(y) = f_k(Py) = 0,$$

and consequently, $f_k(y) = 0$ for all $y \in \overline{Y}$. As $f_k(x_k) \neq 0$, $\overline{Y}$ is a non-trivial closed $\mathcal{J}$-invariant subspace of X. $\square$

According to [18] an operator T on a Hausdorff topological vector space X is weakly quasinilpotent at x_0 if $|f(T^n x_0)|^{\frac{1}{n}} \to 0$ for each $f \in X'$, where X' is the topological dual of X. Z.Ercan and S.Önal in [18] generalized the Theorem 3.1 as follows.

Theorem 3.3: Let $T, A : X \to X$ be two be two non-zero continuous positive operators on a Hausdorff topological vector space X with a Markushevich basis (x_n, f_n). If $TA = AT$ and $T^m A$ is weak quasinilpotent at some positive vector for each m, then T has a nontrivial closed invariant subspace.

Weak quasinilpotence for a family of operators was defined in [14]. A non-empty set $\mathcal{C}$ of linear operators on a topological vector space E is called weakly quasinilpotent at $x_0 \in E$ if $|f(\mathcal{C}^n(x_0))|^{\frac{1}{n}} \to 0$ for each $f \in E'$, where E' is the topological dual and $|f(\mathcal{C}^n(x_0))| = \sup\{|f(T_1 \cdots T_n(x_0))| : T_i \in \mathcal{C}, \;\; i = 1, \cdots, n\}$

The following is a generalization of Proposition 3.2 to topological vector spaces with Markushevich basis.

Proposition 3.4: Let $\mathcal{J}$ be a multiplicative semigroup of positive continuous operators on a Hausdorff topological vector space X with a Markushevich basis (x_n, f_n). If $\mathcal{J}$ is weakly quasinilpotent at some $x_0 > 0$, then $\mathcal{J}$ has a non-trivial closed invariant subspace.

Proof: If $Tx_0 = 0$ for each $T \in \mathcal{J}$, then $\bigcap_{T\in\mathcal{J}} T^{-1}(0)$ is a non-trivial closed subspace that is invariant under $\mathcal{J}$. Thus we can assume that $Tx_0 \neq 0$ for some $T \in \mathcal{J}$. Since $\overline{\text{span}\{x_k\}} = X$, it follows that $Tx_k \neq 0$ for some k. Without loss of generality, we can assume that $0 \leqslant x_k \leqslant x_0$. Consider the projection operator P on X defined by $Px = f_k(x)x_k$. Then $0 \leqslant P \leqslant I$. We claim that $PSTx_k = 0$ for each $S \in \mathcal{J}$. To prove this, let $PSTx_k = \alpha x_k$ for some non-negative scalar α. Since all operators are positive, we have

$$0 \leqslant \alpha^n x_k \leqslant (PST)^n x_k \leqslant (ST)^n x_k \leqslant (ST)^n x_0.$$

Using the positivity of the functional f_k, we have

$$0 \leqslant \alpha^n \leqslant f_k(\alpha^n x_k) \leqslant f_k((ST)^n x_0)$$

and consequently, we get

$$0 \leqslant \alpha \leqslant f_k((ST)^n x_0)^{\frac{1}{n}} \to 0.$$

Hence, $\alpha = f_k\ (PSTx_k) = 0$. Let $Y = \text{span}\ \{STx_k : \quad S \in \mathcal{J}\}$. As $Tx_k \neq 0$, we have $Y \neq \{0\}$. Since $\mathcal{J}$ is a multiplicative semigroup, Y is invariant under $\mathcal{J}$.

Thus, it remains to show $\overline{Y} \neq X$. For each $y \in Y$, we have $f_k(y) = f_k(Py) = 0$ and consequently $f_k(y) = 0$ for all $y \in \overline{Y}$. As $f_k(x_k) \neq 0$, $x_k \notin \overline{Y}$ and $\overline{Y}$ is a non-trivial closed invariant subspace under the semigroup $\mathcal{J}$. □

4. COLLECTIVELY COMPACT SETS

4.1 Introduction

Throughout this chapter, X will denote an infinite dimensional complex Banach space and $B(X)$ will denote the space of all linear bounded operators on X.

In this chapter, we first give some invariant subspace results for collectively compact sets of operators in connection with the joint spectral radius of these sets. We then prove that any collectively compact set M in algΓ satisfies the Berger-Wang formula, where Γ is a complete chain of subspaces of X and algΓ denotes the set of operators that leave all the subspaces in Γ invariant.

Definition 4.1.1: A family $M \subset B(X)$ is called collectively compact if the set $M(U_X) = \{Tx : T \in M, x \in U_X\}$ has compact closure in X, where U_X denotes the closed unit ball in X.

A fairly complete treatment, with applications, of collectively compact sets of linear bounded operators is given in the book [6] by Anselone. We refer to [4], [7], [8], [9], and [15] for more insight into the concept of collectively compact set of operators. For further details on the material in this chapter see [1], [6], [26], [29], and [34], for example.

A set $M \subset B(X)$ is bounded iff $M(U_X)$ is bounded, whereas M is collectively compact iff $M(U_X)$ is precompact. Therefore, every collectively compact set $M \subset B(X)$ is a bounded set of compact operators. The converse is false [6, Example 5.1]. On the other hand, every precompact set of compact operators is collectively compact [6, Proposition 5.3] but the following example shows that the converse is not true in general [6, Example 5.4].

Example 4.1.2: Let M be the set of operators on ℓ_p $(1 \leqslant p \leqslant \infty)$ defined by $T_n x = x_n e_1$ where $x = (x_1, ..., x_n, ...)$ and $e_1 = (1, 0, 0, ...)$. Since MU_X is bounded and one-dimensional, M is collectively compact. But M is not precompact, for $\|T_m - T_n\| = 2^{\frac{1}{p}}$ if $m \neq n$.

Using well-known techniques, we generalize invariant subspace results which are proven for precompact families of compact operators in [34] to collectively compact families of operators. In doing so, we use the following spectral radius formulas for families of operators.

The joint spectral radius $\rho(M)$ of M was introduced by Rota and Strang [31] and is defined by

$$\rho(M) = \limsup_{n\to\infty} \| M^n \|^{1/n} .$$

The similarity to the Gelfand formula for the spectral radius of a single operator underlines the spectral nature of the notion.

The Berger-Wang spectral radius is defined by

$$r(M) = \limsup_{n\to\infty} \sup_{T\in M^n} \{\rho(T)\}^{\frac{1}{n}}.$$

We have

$$r(M) \leqslant \sup_{T\in M^n} \{\rho(T)\}^{\frac{1}{n}} \leqslant \rho(M).$$

The joint local spectral radius of M at a point $x \in X$ is defined by

$$\rho(M,x) = \limsup_{n\to\infty} \|M^n x\|^{\frac{1}{n}}.$$

For a single operator $T \in B(X)$,

$$\rho(T,x) = \limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}}.$$

For $t \geqslant 0$,

$$\mathcal{E}_t(M) := \{x \in X : \ \rho(M,x) \leqslant t\}$$

and similarly,

$$\mathcal{E}_t(T) := \{x \in X : \ \rho(T,x) \leqslant t\}.$$

We need the following results from [34], Lemma 13.1 and Lemma 13.2.

Lemma 4.1.3: Let M be a bounded subset of $B(X)$. Then

a) $\mathcal{E}_t(M)$ is a subspace which is hyperinvariant for M;

b) $\mathcal{E}_t(M) \subset \mathcal{E}_{t^k}(T)$ for all $T \in M^k$ and $k > 0$.

4.2 On Invariant Subspaces of Collectively Compact Sets

A well-known method of proving existence of invariant subspaces for semigroups of operators is the following.

Lemma 4.2.1: Let $\mathcal{S}$ be a multiplicative semigroup of continuous operators on a Banach space and let $\mathcal{S}_0$ be a non-zero semigroup ideal in $\mathcal{S}$. If $\mathcal{S}_0$ has a common non-trivial closed invariant subspace, then $\mathcal{S}$ also has a non-trivial closed invariant subspace.

For a proof of this lemma, we refer to Lemma 10.49 in [1].

For a bounded set M in $B(X)$, we let $\mathrm{LIM}(M)$ denote the set of norm limits of all convergent sequences $\{T_k\}$, where $T_k \in M^{n_k}$ and $n_k \to \infty$. $\mathrm{LIM}(M)$ is a closed semigroup ideal in the closure of $\mathrm{SG}(M)$ cf [34, page 412].

We also need the following lemma from [34, Lemma 6.13].

Lemma 4.2.2: Let M be a bounded set in $B(X)$. Let f be a nonnegative function from the closure of $\mathrm{SG}(M)$ into $\mathbb{R}$ which is continuous at every point of $\mathrm{LIM}(M)$. If $\limsup\limits_{n\to\infty} \sup\limits_{T\in M^n} \{f(T)\}^{\frac{1}{n}} < 1$, then $f(T) = 0$ for all $T \in \mathrm{LIM}(M)$.

Proposition 4.2.3: Let M be a collectively compact set in $B(X)$. If $\mathrm{LIM}(M) \neq \{0\}$ and $r(M) < 1$, then M has an hyperinvariant subspace.

Proof: $\mathrm{LIM}(M)$ is a non-zero semigroup ideal in $\mathrm{SG}(M)$. By taking $\rho(T)$ as $f(T)$ in the preceding lemma and using the continuity of the spectral radius $\rho(T)$ on compact operators, we obtain that $\rho(T) = 0$ for each T in $\mathrm{LIM}(M)$. Thus $\mathrm{LIM}(M)$ consists of quasinilpotent compact operators. As $\mathrm{LIM}(M)$ is a multiplicative semigroup in its own right, it is a Volterra (i.e. compact quasinilpotent) semigroup. It follows from [35, Theorem 4] that $\mathrm{LIM}(M)$ has a non-trivial hyperinvariant subspace. That $\mathrm{SG}(M)$, and in particular M and M' have a common invariant subspace follow from Lemma 4.2.1. □

We will reprove Proposition 4.2.3 in a different way without using Lemma 4.2.2. For this purpose we first start by giving the following definition.

Definition 4.2.4: Let M be a collectively compact set in $B(X)$. We define $lim(M)$ to be the set of all strong limits of convergent sequences $\{T_k\}$, where $\{T_k\}$ is collectively compact, $T_k \in M^{n_k}$ and $n_k \to \infty$.

That is to say $T \in lim(M)$ if and only if $T_k x \to Tx$ for each $x \in X$, $T_k \in M^{n_k}$, and $\{T_k\}$ is collectively compact. Here since

$$T(U_X) \subset \overline{\{T_k x : x \in U_X\}} = \overline{\{T_k\} U_X},$$

it follows that T is compact. In other words, $lim(M)$ consists of compact operators. In what follows, we assume $lim(M) \neq 0$.

Remark 4.2.5: Let $M \subset B(X)$ be a collectively compact set.

(i) If $W \subset X$ is bounded, then MW is precompact.

(ii) M^n is collectively compact for each $n \in \mathbb{N}$.

Lemma 4.2.6: $lim(M)$ is a semigroup ideal in SG(M).

Proof: Let $T \in lim(M)$ and $S \in \mathrm{SG}(M)$. Let $T_k \in M^{n_k}$ and $T_k x \to Tx$ for $x \in X$ as $n_k \to \infty$. Then $ST_k \in M^{j+n_k}$ for some j and $ST_k x \to STx$ for each $x \in X$.

Since $S \in \mathrm{SG}(M)$ is a compact operator, it is immediately seen that $\{ST_k\}$ is a collectively compact sequence.

On the other hand, if $T_k x \to Tx$ for each x and $S \in \mathrm{SG}(M)$, then S is a compact operator and from [6, Proposition 1.8] we have

$$\| (T_k - T)S \| \to 0 \quad \text{as} \quad n_k \to \infty.$$

Thus, we see that not only $T_k Sx \to TSx$ for each $x \in X$ but $\{T_k S, TS\}_k$ is a compact set of compact operators. Hence $\{T_k S, TS\}_k$ is a collectively compact set. It follows that $\{T_k S\}_k$ is collectively compact. $\square$

The following is Theorem 4.8 in [6].

Lemma 4.2.7: Let $T_n, T \in B(X)$. Suppose $T_n x \to Tx$ for each $x \in X$ and $\{T_n - T\}$ is collectively compact. Then for each open set U with $\sigma(T) \subset U$, there exists $N \in \mathbb{N}$ with $\sigma(T_n) \subset U$ for each $n \geqslant N$.

For each $T \in B(X)$ let $\mathcal{F}(T)$ be the family of all complex functions f which are analytic on open, not necessarily connected, domains $\mathcal{D}(f)$ containing the spectrum $\sigma(T)$.

The following is Theorem 4.15 in [6].

Lemma 4.2.8: Let $T_n, T \in B(X)$. Suppose $T_n x \to Tx$ for each $x \in X$, $\{T_n - T\}$ is collectively compact, and $f \in \mathcal{F}(T)$. Then there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $f \in \mathcal{F}(T_n)$ and $f(T_n)x \to f(T)x$ for each $x \in X$. Furthermore, $\{f(T_n) - f(T)\}$ is collectively compact for $n \geqslant N$.

By using the lemmas 4.2.7 and 4.2.8 we obtain the following lemma.

Lemma 4.2.9: Let $T_n, T \in B(X)$. Let $T_n x \to Tx$ for each $x \in X$ and $\{T_n\}$ be collectively compact. If U and V are disjoint open sets with $\sigma(T) \subset U \cup V$ and $\sigma(T) \cap U \neq \phi$, then there exists $N \in \mathbb{N}$ such that $\sigma(T_n) \cap U \neq \phi$ for all $n \geqslant N$.

Proof: Let us observe that the family $\{T_n - T\}$ is collectively compact. Let f be the analytic function defined on $U \cup V$ as $f(\lambda) \equiv 1$ on U and $f(\lambda) \equiv 0$ on V. By Lemma 4.2.7, $\sigma(T_n) \subset U \cup V$ for all $n \geqslant N$ for some $N \in \mathbb{N}$. Assume that the claim is not true. Then for each N, there exists $n \geqslant N$ such that $\sigma(T_n) \subset V$. In this way, we construct a subsequence of operators (T_{n_k}) with the property that $\sigma(T_{n_k}) \subset V$ for each k. On the other hand, as $f(T_n)x \to f(T)x$ from Lemma 4.2.8, we have $f(T_{n_k})x \to f(T)x$. As $f(T_{n_k}) = 0$ for each k, $f(T)x = 0$ for each $x \in X$. By the spectral theorem $\sigma(f(T)) = f(\sigma(T))$ and 1 must be in the spectrum of the operator $f(T)$, $f(T)$ can not be the zero operator. This contradiction yields $\sigma(T_n) \cap U \neq \emptyset$ for all $n \geqslant N$ for some N. $\square$

Lemma 4.2.10: Let M be a collectively compact set in $B(X)$. If $r(M) < 1$, then $\rho(T) = 0$ for all T in $\mathit{lim}(M)$.

Proof: Let $r(M) = \limsup_{n\to\infty} \sup_{S \in M^n} \{\rho(S)\}^{\frac{1}{n}} = t < 1$. Let $T \in \mathit{lim}(M)$ and let $\{T_n\}$ be a collectively compact sequence in SG(M) with $T_n x \to Tx$ for each x. It follows that T is a compact operator and $\{T_n - T\}$ is collectively compact. Thus for a given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{S \in M^n} \rho(S)^{\frac{1}{n}} < t + \epsilon < 1 \quad \text{for all} \quad n \geqslant n_0.$$

Hence for each $T_n \in M^{n_k}$,

$$\rho(T_n) < (t + \epsilon)^n < 1 \quad \text{for} \quad n \geqslant n_0.$$

It follows that $\rho(T_n) \to 0$.

We show next that $\rho(T_n) \to \rho(T)$. Consider the open ball $B(0, \rho(T))$ centered at 0 with radius $\rho(T)$. Then there exists an $N_1 \in \mathbb{N}$ such that $\sigma(T_n) \subset B(0, \rho(T))$ for all $n \geqslant N_1$. It follows that $\rho(T_n) \leqslant \rho(T)$ for $n \geqslant N_1$. Hence

$$\limsup_{n\to\infty} \rho(T_n) \leqslant \rho(T).$$

On the other hand, there is only a finite number of points $\lambda_1, \cdots, \lambda_m$ of the spectrum $\sigma(T)$ that lie outside $B(0, \rho(T))$, each of which satisfies $|\lambda_i| = \rho(T)$, $i = 1, \cdots, m$. As each of the points λ_i $(i = 1, \cdots, m)$ is an isolated point of the spectrum $\sigma(T)$, $\{\lambda_1, \cdots, \lambda_m\}$ is an open subset of $\sigma(T)$ and $\sigma(T) \cap \{\lambda_1, \cdots, \lambda_m\} \neq \phi$. It follows that for some i $(i = 1, \cdots, m)$, λ_i is an element of $\sigma(T_n)$ for all $n \geqslant N_2$ for some $N_2 \in \mathbb{N}$. Thus,

$$\rho(T) = |\lambda_i| \leqslant \rho(T_n) \quad \text{for all} \quad n \geqslant N_2$$

and so

$$\rho(T) \leqslant \liminf_{n\to\infty} \rho(T_n).$$

Therefore, we have $\lim_{n\to\infty} \rho(T_n) = \rho(T)$ and $\rho(T) = 0$. □

With the help of the preceding auxiliary results the proof of the following proposition is immediate and is omitted.

Proposition 4.2.11: Let M be a collectively compact set in $B(X)$. If $r(M) < 1$, then $\mathrm{SG}(M)$ has an hyperinvariant subspace.

Corollary 13.5 in [34] states that for a locally finitely quasinilpotent subset $\mathcal{C}$ of $B(X)$, if a closed subalgebra $\mathcal{A}$ generated by $\mathcal{C}$ and $\mathcal{C}'$ contains a non-zero compact operator, then $\mathcal{C}$ has a non-trivial hyperinvariant subspace. So, any locally quasinilpotent collectively compact set in $B(X)$ has a non-trivial closed hyperinvariant subspace. Therefore, in what follows we assume that $\rho(M, x) \neq 0$ for each $x \in X$.

Proposition 4.2.12: Let $M \subset B(X)$ be collectively compact. If there exists some non-zero $x \in X$ such that $\rho(M, x) < \sup_{T\in M^n} \rho(T)^{\frac{1}{n}}$, then M has an hyperinvariant subspace.

Proof: The assumption yields an operator $T \in M^k$ such that $\rho(M, x) \leqslant \rho(T)^{\frac{1}{k}}$ for some k. Let $\rho(M, x) = t$. Then, $t^k < \rho(T)$ and we have $\mathcal{E}_t(M) \subseteq \mathcal{E}_{t^k}(T)$ for this particular t. Also, $\mathcal{E}_t(M) \neq 0$ as $x \in \mathcal{E}_t(M)$. We claim $\overline{\mathcal{E}_{t^k}(T)} \neq X$.

For any circle centered at 0 with radius r there are only finite number points of $\sigma(T)$ that lie outside the circle. Choosing the radius r, appropriately, as $t^k < r < \rho(T)$, we can make certain that the circle $\{z : |z| = r\}$ lies entirely in the resolvent of T. Let σ be the part of the spectrum that lies in the circle and let $\sigma' = \sigma(T) \setminus \sigma$. Then σ is a spectral set. Let $P_\sigma(T)$ be the corresponding spectral projection. Then $\mathcal{E}_{t^k}(T)$ is contained in the range of $P_\sigma(T)$ as the range of $P_\sigma(T)$ can be identified as $\{x \in X : \frac{T^n x}{r^n} \to 0\}$ cf. Exercise 6.4.14 in [1]. Taking closures we see that $\overline{\mathcal{E}_{t^k}(T)} \subset \overline{\mathrm{Ran} P_\sigma(T)} = \mathrm{Ran} P_\sigma(T)$. On the other hand, $r < \rho(T)$ and $P_\sigma(T) \neq I$. Thus $\overline{\mathcal{E}_t(M)} \neq X$ and M has an hyperinvariant subspace. □

Our next result is a simple consequence of Theorem 13.10 in [34] which states that if SG(M) consists of operators with countable spectra and $\rho(M,x) < r(M)$ for some non-zero $x \in X$, then M has a nontrivial hyperinvariant subspace.

Proposition 4.2.13: Let M be a collectively compact family in $B(X)$ with $\rho(M,x) < r(M)$ for some non-zero x in X. Then M has a nontrivial hyperinvariant subspace.

A Theorem due to Yu. V. Turovskii in [35, Theorem 2] states the following:

Theorem 4.2.14: Let M be a precompact set of compact operators with $\rho(M) = 1$. If SG(M) is not bounded then M has a nontrivial hyperinvariant subspace.

The following theorem proved by V.S. Shulman and Yu.V. Turovskii in [34, Theorem 6.10] is an extension of the previous theorem.

Theorem 4.2.15: Let M be a precompact set of bounded operators with $\rho_e(M) < \rho(M) = 1$. If $SG(M)$ is not bounded then M has a nontrivial hyperinvariant subspace.

We aim to generalize Turovskii's Theorem to collectively compact sets of operators. For this we need the following observation. Given a bounded set M of bounded operators on X, SG$(t^{-1}M)$ is bounded if $t > \rho(M)$. Indeed, in this case $\rho(t^{-1}M) = t^{-1}\rho(M) < 1$ and, therefore, $\|(t^{-1}M)^n\| < 1$ for all sufficiently large $n > 0$.

Proposition 4.2.16: Let M be a collectively compact set of operators with $\rho(M) = 1$. If $SG(M)$ is not bounded, then M has a nontrivial hyperinvariant subspace.

Proof Let (t_n) be a sequence of reals such that $t_n \to 1^+$. Since $SG(t_n^{-1}M)$ is bounded, we may define functions φ_n on X as follows:

$$\varphi_n(x) = s_n^{-1} \, \|SG_1(t_n^{-1}M)x\|$$

for all $x \in X$, where $s_n = \|SG_1(t_n^{-1}M)\|$.

All φ_n are norms on X equivalent to $\|\cdot\|$, and $\varphi_n(Tx) \leqslant t_n\varphi_n(x)$ for all $T \in M$ and all $x \in X$. Let φ be the function on X defined by

$$\varphi(x) = \limsup_{n\to\infty} \varphi_n(x) \quad \text{for all} \quad x \in X.$$

We show that φ is a non-zero continuous seminorm whose kernel, $\mathrm{Ker}\varphi = \{x \in X : \varphi(x) = 0\}$ is non-zero and $\mathrm{Ker}\varphi$ is a hyperinvariant subspace of M.

φ is a seminorm on X. Since $\varphi_n(x) \leqslant \|x\|$ for all $x \in X$, φ is continuous on X and, therefore, $\mathrm{Ker}\varphi$ is a closed subspace of X. Since,

$$\varphi(Tx) = \limsup_{n\to\infty} \varphi_n(Tx) \leqslant \limsup_{n\to\infty} t_n\varphi_n(x) = \varphi(x)$$

and

$$\varphi(Sx) = \limsup_{n\to\infty} \varphi_n(Sx) \leqslant \|S\| \limsup_{n\to\infty} \varphi_n(x) = \|S\|\varphi(x)$$

for all $x \in X$, all $T \in M$ and all $S \in M'$, the commutant of M. Thus $\mathrm{Ker}\varphi$ is a hyperinvariant subspace under M.

We first show that $\mathrm{Ker}\varphi \neq X$; that is $\varphi \neq 0$.

Choose $x_n \in X$ such that $\varphi_n(x_n) > t_n^{-1}$ and $\|x_n\| = 1$. By the definition of $\varphi_n(x_n)$, there exists a sequence $\{T_n\}$ of elements of $SG_1(t_n^{-1}M)$ such that

$$s_n^{-1}\|T_n x_n\| > t_n^{-1} \quad \text{for all} \quad n.$$

As $SG(M)$ is unbounded, there exits n_0 such that $T_n \neq I$ for all $n > n_0$. So one may suppose $T_n = Q_n S_n$ for all n, where $S_n \in t_n^{-1}M$ and $Q_n \in SG_1(t_n^{-1}M)$. Consider the sequence $\{t_n S_n\}$ in M. Since M is collectively compact, each sequence in M is also collectively compact and, therefore, the sequence $\{t_n S_n x_n\}$

is relatively compact. Therefore, it has a convergent subsequence which we denote by $\{t_{n_k} S_{n_k} x_{n_k}\}$.

Suppose $t_{n_k} S_{n_k} x_{n_k} \to y \in X$. Then $\|S_{n_k} x_{n_k} - y\| \to 0$. Since

$$\varphi_n(S_n x_n - y) \leqslant \|S_n x_n - y\|$$

for all n and

$$\varphi_{n_k}(y) \geqslant \varphi_{n_k}(S_{n_k} x_{n_k}) - \varphi_{n_k}(S_{n_k} x_{n_k} - y) \geqslant \varphi_{n_k}(S_{n_k} x_{n_k}) - \|S_{n_k} x_{n_k} - y\|$$

for all n_k, we have

$$\varphi(y) = \limsup_{n\to\infty} \varphi_n(y) \geqslant \limsup_{k\to\infty} \varphi_{n_k}(S_{n_k} x_{n_k}).$$

On the other hand, for each n,

$$t_n^{-1} < s_n^{-1} \|Q_n S_n x_n\| \leqslant s_n^{-1} \|SG_1(t_n^{-1} M) S_n x_n\| = \varphi_n(S_n x_n)$$

which yields that $\varphi(y) \geqslant 1$. Thus $\mathrm{Ker}\varphi \neq X$ and $\varphi \neq 0$.

We now show that $\mathrm{Ker}\varphi \neq 0$.

An operator $T \in M^n$ is called leading for M if $\|T\| \geqslant \|\bigcup_{i=1}^{n-1} M^i\|$. Since $SG(M)$ is not bounded, $\{\|\bigcup_{i=1}^{n} M^i\|\}_{n=1}^{\infty}$ is not bounded. Therefore, there exists an increasing sequence $\{m_k\}$ such that $\|M^{m_k}\| > \|\bigcup_{i=1}^{m_k-1} M^i\|$ and, $\|\bigcup_{i=1}^{m_k-1} M^i\| \to \infty$. Hence, there exists a sequence $\{T_k\}$ of operators, $T_k \in M^{m_k}$, that are leading for M with $\|T_k\| \to \infty$.

Let $\alpha_k = \|T_k\|^{-1}$. We choose $x_k \in X$ with $\|\alpha_k T_k x_k\| > t_k^{-1}$ for all k. Since $\rho(M) = \lim_{n\to\infty} \|M^n\|^{\frac{1}{n}} = 1$, then $\|M^n\| = 1$ for all n. Hence, we can write each of the operators $\alpha_k T_k$ in the form $P_k(\alpha_k Q_k)$ where $P_k \in M$, $\alpha_k Q_k \in SG_1(M)$ and the sequence $\{\alpha_k Q_k\}$ is bounded.

The sequence $\{\alpha_k Q_k x_k\}$ is a bounded sequence. Therefore, the sequence $\{P_k \alpha_k Q_k x_k\} = \{\alpha_k T_k x_k\}$ has a convergent subsequence as $\{P_k\}$ is collectively compact. Let $P_{k_r} \alpha_{k_r} Q_{k_r} x_{k_r} \to y_0$ as $r \to \infty$. That $y_0 \neq 0$ follows from the fact that $\|\alpha_{k_r} T_{k_r} x_{k_r}\| > t_{k_r}^{-1}$ for each r.

On the other hand, since $\varphi(T_{k_r}x_{k_r}) \leqslant \varphi(x_{k_r})$ and $\varphi(x_{k_r}) \leqslant 1$, we get

$$\varphi(y_0) = \lim_{r\to\infty} \varphi(\alpha_{k_r} T_{k_r} x_{k_r}) \leqslant \limsup_{r\to\infty} \alpha_{k_r}\varphi(x_{k_r}) = \limsup_{r\to\infty} \alpha_{k_r} = 0.$$

That is, $\varphi(y_0) = 0$ and so $y_0 \in \mathrm{Ker}\varphi$. Since $y_0 \neq 0$, we obtain $\mathrm{Ker}\varphi \neq 0$. □

4.3 Berger-Wang Formula

It was shown in [12] that $\rho(M) = r(M)$ if X is finite dimensional linear space. This formula is called the Berger-Wang formula. Furthermore, it was shown in [34] that the Berger-Wang formula is valid for precompact sets of compact operators and for precompact sets of essentially scalar operators on an infinite dimensional Banach space. We note that the Berger-Wang formula does not extend to arbitrary, even two-element, sets of bounded operators. This was shown by an example in [30], which was based on an example in [19]. See also [21, Problem 1 J]. In this section we will show, in a special case, that any collectively compact set of operators satisfies the Berger-Wang formula. To do this, we first recall some definitions and introduce notation.

A chain of subspaces of X is a linearly ordered by inclusion set of subspaces. Γ is a complete chain if for all $\Theta \subset \Gamma$, $\sup\Theta = \overline{\bigcup_{Y\in\Theta} Y}$, $\inf\Theta = \bigcap_{Y\in\Theta} Y$, $\{0\}$, and X are in Γ. A maximal subspace chain is a chain of subspaces that is not properly contained in any other chain of subspaces. If $\mathcal{F}$ is any set of operators, then lat$\mathcal{F}$ is the collection of all subspaces that are invariant under all the operators in $\mathcal{F}$.

Let Γ be a complete chain of closed subspaces of X.
For any $Z \in \Gamma$, let $Z_- = \sup\{Y \in \Gamma : Y \subset Z, Y \neq Z\}$, and let $\{0\}_- = 0$. If $Z_- \neq Z$, then the pair (Z_-, Z) is called a gap of Γ and the quotient Z/Z_- is called a gap quotient of Γ. The algebra of all operators preserving invariant all subspaces in Γ is algΓ. If $T \in \mathrm{alg}\Gamma$ and $V = Z/Z_-$, then the operator induced by T in V is denoted by $T|\,V$. Also, $T|\,Z$ means the restriction of $T \in \mathrm{alg}\Gamma$ to $Z \in \Gamma$. The same notation will be used for sets of operators in algΓ. We define the function gap_Γ on Γ as follows: $\mathrm{gap}_\Gamma(Z) = Z/Z_-$ for all $Z \in \Gamma$.

It is well-known that a chain of subspaces of X is maximal as a subspace chain if and only if it contains $\{0\}$ and X, it is complete, and $\dim Z/Z_- \leqslant 1$ for any Z in the chain; see [26, Theorem 7.1.9].

The following is Corollary 4.3 in [34].

Lemma 4.3.1: Let F be a finite chain of closed subspaces with $\{0\}, X \in F$ and $M \subset B(X)$ be a bounded set in alg$F$. Then

$$\rho(M) = \max\{\rho(M|\ V) : V \in \text{gap}(F)\}.$$

Let Γ denote a complete chain of closed subspaces of X.

Definition 4.3.2: Let $\text{alg}_t\Gamma$ be the set of all operators $T \in \text{alg}\Gamma$ satisfying $\text{dist}(Tz, Z_-) \leqslant t\|z\|$ for any gap (Z_-, Z) in Γ and all $z \in Z$. A chain F of closed subspaces of X is called t-chain for $M \subset B(X)$ if $M \subset \text{alg}_t F$ and $\{0\}, X \in F$.

The following is Lemma 4.6 in [34].

Lemma 4.3.3: Any finite set M of compact operators in $\text{alg}_t\Gamma$ has a finite $(t+\epsilon)$-chain for any $\epsilon > 0$.

Theorem 4.7 of [34] states that if M is a precompact set of compact operators in $\text{alg}_t\Gamma$, then $\rho(M) \leqslant t$. Therefore, it is natural to ask whether this result remains true for collectively compact families. The following extends this theorem to collectively compact set of operators in $\text{alg}_t\Gamma$.

Proposition 4.3.4: Let M be a collectively compact set of operators in $\text{alg}_t\Gamma$. Then $\rho(M) \leqslant t$.

Proof: Let $\epsilon > 0$ be given. Let $M_0 \subset M$ be a finite set such that if $T \in M$, then there exists some $T_i \in M_0$, $i = 1, ..., n$ and some y_j, $j = 1, ..., m$, $\|y_j\| \leqslant 1$ such that $\|Tx - T_i y_j\| \leqslant \epsilon$ for x, $\|x\| \leqslant 1$ for some i and j.

Since M_0 is a finite set of compact operators in $\text{alg}_t\Gamma$, it follows from Lemma 4.3.3 that it has a finite $(t+\epsilon)$-chain F. That is to say $\|T|\ \text{gap}_F(Z)\| < t+\epsilon$ for all $T \in M_0$ and $Z \in F$. We wish to show $\|T|\ \text{gap}_F(Z)\| < t + 2\epsilon$ for all $T \in M$.

Let $T \in M$ and $\widehat{x}$ be an arbitrary element in the closed unit ball of $V = Z/Z_-$ where $Z \in F$. Then $\widehat{x} = y$ for some y in Z with $\|y\| \leqslant 1$. Then

$$\begin{aligned} \|T|\ V\widehat{x}\| &\leqslant \|T|\ V\widehat{x} - T_i|\ V\widehat{y_j}\| + \|T_i|\ V\widehat{y_j}\| \\ &\leqslant \|Tx - T_i y_j\| + \|T_i y_j\| \\ &< t + 2\epsilon, \end{aligned}$$

where $T_i y_j$ are chosen in accordance with collective compactness of M. Since the preceding inequality is true for all $\widehat{x}$ in Z/Z_- with $\|\widehat{x}\| \leqslant 1$, we have

$$\|T|\ V\| < t + 2\epsilon$$

for all $T \in M$. Thus

$$\|S|\ \mathrm{gap}(Z)\| < t + 2\epsilon$$

for all $S \in M$. Therefore, we have

$$\rho(M|\ \mathrm{gap}_F(Z)) < t + 2\epsilon$$

for all $Z \in F$. However, for a finite chain of closed subspaces and a bounded set M in alg$F$, $\rho(M) = \max\{\rho(M|\ V) : V \in \mathrm{gap}(F)\}$. Thus, $\rho(M) < t + 2\epsilon$. Letting $\epsilon \to 0$, we obtain $\rho(M) \leqslant t$. □

Recall that a family M of operators is triangularizable if there exists a maximal chain of subspaces of X each of which is invariant for M.

The proof of the following corollary is exactly the same as that of Corollary 4.8 in [34].

Corollary 4.3.5: Let $M \subset B(X)$ be a collectively compact triangularizable set of operators. Then $\rho(M) = \sup\{\rho(T) : T \in M\}$.

As usual, Γ denotes a complete subspace chain.

Proposition 4.3.6: Let M be a collectively compact set in algΓ and $\epsilon > 0$. Then there exists only a finite number of gaps (Z_-, Z) of the chain Γ such that $\|M|\ \mathrm{gap}(Z)\| > \epsilon$.

Proof: Suppose, to the contrary, that the set F of all such gaps is infinite. For every gap (Z_-, Z) in F, pick $T \in M$ and $x \in U_Z$ such that $\|Tx + Z_-\| > \epsilon$. Let W be the set of such vectors Tx when (Z_-, Z) runs over all gaps from F. Let Tx and Sy be any two vectors in W corresponding to some different gaps (U_-, U) and (V_-, V). As Γ is complete, we may suppose $V \subset U_-$. Then $Sy \in U_-$ and $\|Tx - Sy\| \geqslant \|Tx + U_-\| > \epsilon$. Since F is infinite, W is not precompact. On the other hand, $W \subset MU_X$. Thus, W is precompact and we obtain a contradiction. □

The following was proved for precompact sets of compact operators in algΓ. See Theorem 5.4 in [34].

Proposition 4.3.7: Let M be a collectively compact set of operators in algΓ. Then,

$$\rho(M) = \rho(M|\ \Gamma) = \max\{\rho(M|\ \text{gap}Z) : Z \in \Gamma\}.$$

Proof: Let $\rho(M) > 0$ and $t = \frac{\rho(M)}{2}$. Let $F_0 \subset \Gamma$ be the family of subspaces Z_-, Z for all gaps (Z_-, Z) of Γ such that $\|M|\ \text{gap}_\Gamma(Z)\| > t$. Put $F = F_0 \cup \{\{0\}, X\}$. It follows from Proposition 4.3.6 that F is finite subchain of Γ. By Lemma 4.3.1 it follows that

$$\rho(M) = \max\{\rho(M|\ \text{gap}_F(U)) : U \in F\}.$$

Let (V, U) be a gap of F which is not a gap of Γ. It suffices to show $\rho(M|\ (U/V)) < \rho(M)$. Let

$$\Gamma_0 = \{Z/V : Z \in \Gamma, V \subset Z \subset U\}.$$

Let $(Z_1/V, Z_2/V)$ be any gap of the chain Γ_0. Then (Z_1, Z_2) is a gap of Γ and

$$\|M|\ (Z_2/Z_1)\| \leqslant t.$$

Identifying Z_2/Z_1 with $(Z_2/V)/(Z_1/V)$, we observe that

$$\|M|\ ((Z_2/V)/(Z_1/V))\| \leqslant t.$$

Thus $M|\ (U/V) \subset \text{alg}_t\Gamma_0$ and $M|\ (U/V)$ is collectively compact. Thus by Proposition 4.3.4, $\rho(M|\ (U/V)) \leqslant t$ and so $\rho(M|\ (U/V)) < \rho(M)$. □

Proposition 4.3.8: Let $M \subset B(X)$ be a collectively compact set of operators in algΓ where Γ is a complete subspace chain. Then $\rho(M) = r(M)$.

Proof: It suffices to prove $\rho(M) \leqslant r(M)$ whenever $\rho(M) > 0$. By Proposition 4.3.7, there exits a $Z \in \Gamma$ such that $\rho(M) = \rho(M|\ \text{gap}Z)$. Since gap$Z$ is one dimensional and $\rho(M) = r(M)$ for all bounded subset of $B(X)$ if X is a finite dimensional linear space, it follows that $\rho(M|\ \text{gap}Z) = r(M|\ \text{gap}Z)$. Since $r(M|\ \text{gap}Z) \leqslant r(M)$ by Lemma 7.5 in [34], we obtain the result. □

5. RESULTS AND DISCUSSION

In this thesis, the invariant subspace problem is studied for certain families of operators on Banach spaces and Banach lattices.

In Chapter 2, we prove that every non-zero locally quasinilpotent compact-friendly operator on a Banach lattice with separating orthomorphisms has a non-trivial closed invariant ideal. We then generalize it by using the concept of compact-friendliness as follows: Every locally finitely quasinilpotent family of positive operators on a Banach lattice with separating orthomorphisms, whose commutant contains a positive operator which dominates an operator which is dominated by a compact positive operator, has a common non-trivial closed invariant ideal.

In Chapter 3, we prove that a locally finitely quasinilpotent multiplicative semigroup of positive continuous operators on a Banach space with a Schauder basis has a non-trivial closed invariant subspace. We then generalize our result to topological vector spaces with Markushevich basis by using the notion of weakly quasinilpotence.

In Chapter 4, we generalize invariant subspace results which are proven for precompact families of compact operators in [34] to collectively compact families of operators. We first show that if M is a collectively compact family in $B(X)$ and the Berger-Wang spectral radius $r(M)$ is less than one, then the multiplicative semigroup $\mathrm{SG}(M)$ generated by M has an invariant subspace. Another results in this direction are the ones which yield a common invariant subspace for a collectively compact family M of operators if $\rho(M,x) < \sup_{T\in M^n} \rho(T)^{\frac{1}{n}}$ and $\rho(M,x) < r(M)$ for some non-zero $x \in X$. Moreover, if the joint spectral radius $\rho(M)$ is exactly one and $\mathrm{SG}(M)$ is not bounded, we then show that M has a non-trivial hyperinvariant subspace. In the final part of this chapter, we consider a complete chain Γ of closed subspaces of X and show that if M is a collectively compact family in $\mathrm{alg}_t\Gamma$ then we have $\rho(M) \leqslant t$. As a result of this, for triangularizable collectively compact sets M of operators we prove that $\rho(M) = \sup\{\rho(T) : T \in M\}$. We also show that if M is a collectively family in algΓ and $\epsilon > 0$, then there exists only a finite number of gaps (Z_-, Z) of the chain Γ such that $\|M| \text{ gap}(Z)\| > \epsilon$. By using this result we obtain that if M is a collectively compact family in algΓ, then we have $\rho(M) = \rho(M| \ \Gamma) = \max\{\rho(M| \text{ gap}Z) : Z \in \Gamma\}$. We finally show that the Berger-Wang formula $\rho(M) = r(M)$ holds for a collectively compact family M in algΓ where Γ is a complete chain of subspaces. In view of the papers [4], [16], and [22], the results obtained in this chapter may be adapted to collectively compact sets of positive operators on Banach lattices.

BIBLIOGRAPHY

[1] **Abramovich, Y.A. and Aliprantis, C.D.**, 2002. An invitation to operator theory, Graduate Studies in Math., Vol. 50, Amer. Math. Soc.

[2] **Abramovich, Y.A., Aliprantis, C.D., and Burkinshaw, O.**, 1994. Invariant subspace theorems for positive operators, *J. Funct. Anal.*, **124**, 95-111.

[3] **Abramovich, Y.A., Aliprantis, C.D., and Burkinshaw, O.**, 1995. Invariant subspaces for positive operators acting on a Banach space with basis, *Proc. Amer. Math. Soc.*, **123**, 1773-1777.

[4] **Alpay, Ş.**, 1989. On collectively compact positive operators, *Rev. Roumaine Math. Pures Appl.* **34**, 9, 779-791.

[5] **Alpay, Ş. and Mısırlıoğlu, T.**, 2005. Invariant subspace theorems for families on a Banach lattice, preprint.

[6] **Anselone, P.M.**, 1971. Collectively compact operator approximation theory and applications to integral equations, Prentice-Hall Inc.

[7] **Anselone, P.M. and Moore, R.H.**, 1964. Approximate solutions of integral and operator equations, *J. Math. Anal. Appl.*, **9**, 268-277.

[8] **Anselone, P.M. and Palmer, T.W.**, 1968. Collectively compact sets of linear operators, *Pacific J. Math.*, **25**, 417-422.

[9] **Anselone, P.M. and Palmer, T.W.**, 1968. Spectral analysis of collectively compact, strongly convergent operator sequences, *Pacific J. Math.*, **25**, 423-431.

[10] **Aronszajn, N. and Smith, K.T.**, 1954. Invariant subspaces of completely continuous operators, *Ann. of Math.*, **60**, 345-350.

[11] **Aupetit, B.**, 1991. A Primer on Spectral Theory, Springer.

[12] **Berger, M.A. and Wang, Y.**, 1992. Bounded semigroups of matrices, *Linear Algebra Appl.*, **166**, 21-27.

[13] **Brown, S.W.**, 1978. Some invariant subspaces for subnormal operators, *Integral Equations Operator Theory*, **1**, 310-333.

[14] **Çaglar, M.**, 2005. Invariant subspaces of positive operators on Riesz spaces and observation on $CD_0(K)$-spaces, *Ph.D. Thesis*, Middle East Technical University.

[15] **DePree, J.D. and Higgins, J.A.**, 1970. Collectively compact sets of linear operators, *Math. Z.*, **115**, 366-370.

[16] **Drnovsek, R.**, 2001. Common invariant subspaces for collections of operators, *Integr. Equ. Oper. Theory.*, **39**, 253-266.

[17] **Enflo, P.**, 1975-1978. On the invariant subspace problem for Banach spaces, Seminaire Maurey-Schwarz; *Acta Math.*, **158**, (1987) 213-313.

[18] **Ercan, Z. and Önal, S.**, 2004. Invariant subspaces for positive operators acting on a Banach space with Markushevich basis, *Positivity* **8**, 123-126.

[19] **Guinand, P.S.**, 1982. On quasinilpotent semigroups of operators, *Proc. Amer. Math. Soc.*, **86**, 3, 485-486.

[20] **Hadwin, D.W., Nordgren, E., Radjavi, H., and Rosenthal, P.**, 1980. An operator not satisfying Lomonosov's hypothesis, *J. Funct. Anal.*, **38**, 410-415.

[21] **Halmos, P.**, 1991. Problems for mathematicians, young and old, Math. Assoc. Amer., Washington DC.

[22] **Jahandideh, M.T.**, 1997. On the ideal-triangularizability of positive operators on Banach lattices, *Proc. Amer. Math. Soc.*, **125**, 9, 2661-2670.

[23] **Lomonosov, V**, 1973. Invariant subspaces for operators commuting with compact operators, *Funct. Anal. Appl.*, **7**, 213-214.

[24] **de Pagter, B.**, 1981. f-algebras and orthomorphisms, *Ph.D. Thesis*, University of Leiden.

[25] **de Pagter, B.**, 1986. Irreducible compact operators, *Math. Z.*, **192**, 149-153.

[26] **Radjavi, H. and Rosenthal, P.**, 2000. Simultaneous Triangularization, Springer.

[27] **Read, C.J.**, 1985. A solution to the invariant subspace problem on the space ℓ_1, *Bull. London Math. Soc.*, **17**, 305-317.

[28] **Read, A.W.**, 1997. Quasinilpotent operators and the invariant subspace problem, *J. London Math. Soc.*(2) **56**, 595-606.

[29] **Riesz, F. and Sz.-Nagy, B.**, 1990. Functional Analysis, Dover.

[30] **Rosenthal, P. and Soltysiak, A.**, 1995. Formulas for the joint spectral radius of non-commuting Banach algebra elements, *Proc. Amer. Math. Soc.*, **123**, 9, 2705-2708.

[31] **Rota, G.C. and Strang, W.G.**, 1960. A note on the joint spectral radius, *Indag. Math.*, **22**, 379-381.

[32] **Singer, I.**, 1970. Bases in Banach spaces I, Springer.

[33] **Shulman, V.S.**, 1984. On invariant subspaces of Volterra operators, *Funk. Anal. i Prilozen*, 84-85.

[34] **Shulman, V.S. and Turovskii, Yu.V.**, 2000. Joint spectral radius, operator semigroups and a problem of W.Wojtynski, *J. Funct. Anal.*, **177**, 383-441.

[35] **Turovskii, Yu.V.**, 1999. Volterra semigroups have invariant subspaces, *J. Funct. Anal.* **162**, 313-322.

[36] **Wickstead, A.W.**, 1980. Extensions of orthomorphisms, *J. Austral. Math. Soc.* **29**, 87-98.

VITA

Remzi Tunç Mısırlıoğlu was born on December 20, 1970 in Urfa, Türkiye. After graduating in 1989 from the High School of Bahçelievler in İstanbul, he entered the Department of Mathematics in Faculty of Science at İstanbul University. In 1996, he started the MSc program in Mathematics at the Institute of Science and Technology of İstanbul Technical University (ITU) after one-year English Preparatory School at ITU. Once he had defended his Master Thesis in 1999, he entered the PhD program in Mathematics at the same Institute. Also, he has been working as a research assistant in the Department of Mathematics at ITU since 1998.