İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

CR-SUBMANIFOLDS OF LOCALLY CONFORMAL KAEHLER MANIFOLDS

M.Sc. Thesis by
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Programme : Mathematical Engineering

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JUNE 2010

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Date of submission : **05 May 2010**
Date of defence examination: **07 June 2010**

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JUNE 2010

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

LOKAL OLARAK KONFORM KAEHLER MANİFOLDLARIN CR-ALTMANİFOLDLARI

YÜKSEK LİSANS TEZİ

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Tezin Enstitüye Verildiği Tarih : 05 Mayıs 2010
Tezin Savunulduğu Tarih : 07 Haziran 2010

Tez Danışmanı : Prof. Dr. Zerrin ŞENTÜRK (İTÜ)
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HAZİRAN 2010

ACKNOWLEDGEMENTS

I would like to express my hearty gratitude to my advisor Prof. Dr. Zerrin ŞENTÜRK. Without her guidance this study wouldn't be as neat as it is. Thanks for the time that you spend to examine my notes meticulously.

May 2010 Selçuk KAYACAN

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SUMMARY

In this thesis, CR-submanifolds of locally conformal Kaehler manifolds are presented.

Taking into account the theorem of Frobenius it is proved that the necessary and sufficient conditions for the integrability of the maximal holomorphic distribution $\mathcal{D}$ and the complementary orthogonal distribution $\mathcal{D}^{\perp}$ on a CR-submanifold M of an almost Hermitian manifold N. We also showed that, if N is a Hermitian manifold, then M is a CR-manifold which justifies the name CR-submanifold.

It is well known that an Hermitian manifold N is a Kaehler manifold if and only if $d\Omega = 0$, whereby Ω is the fundamental 2-form of N. Let us consider a slightly larger class of Hermitian manifolds, namely those for which $d\Omega = \Omega \wedge \alpha$ for some 1-form α. It is proved that if N is a Hermitian manifold with $d\Omega = \Omega \wedge \alpha$, then in order that the submanifold M be a CR-submanifold it is necessary that the totally real distribution $\mathcal{D}^{\perp}$ be integrable.

Finally, if α is a closed 1-form we call these manifolds locally conformal Kaehler manifolds. In this study, we set the conditions for the holomorphic distribution $\mathcal{D}$ to be integrable provided that the ambient space N is a locally conformal Kaehler one.

LOKAL OLARAK KONFORM KAEHLER MANİFOLDLARIN

CR-ALTMANİFOLDLARI

ÖZET

Bu tez çalışmasında lokal olarak konform Kaehler manifoldların CR-altmanifoldları sunulmuştur.

Frobenius Teoremi göz önüne alınarak hemen hemen Hermit bir N manifoldunun bir CR-altmanifoldu M nin maksimal holomorfik distribüsyonu $\mathcal{D}$ ve $\mathcal{D}$ nin tamamlayıcı dik distribüsyonu $\mathcal{D}^{\perp}$ ün integrallenebilmesi için gerek ve yeter koşullar ispatlanmıştır. Eğer N bir Hermit manifold ise, bu taktirde M nin bir CR-manifold olduğu gösterilmiştir ki, bu Teorem CR-altmanifoldunun CR-manifold olarak isimlendirilebileceğini gösterir.

Ω bir N Hermit manifoldunun temel 2-formunu göstermek üzere, N nin bir Kaehler manifold olması için gerek ve yeter şartın $d\Omega = 0$ olduğu bilinmektedir. Hermit manifoldların daha geniş bir sınıfını göz önüne alalım. Diğer bir deyişle, α, bir 1-form olmak üzere $d\Omega = \Omega \wedge \alpha$ dır. Eğer N, $d\Omega = \Omega \wedge \alpha$ koşulunu sağlayan bir Hermit manifold ise, M altmanifoldunun bir CR-altmanifold olması için gerek şartın total olarak reel distribüsyon $\mathcal{D}^{\perp}$ ün integrallenebilmesi olduğu ispatlanmıştır.

Son olarak, eğer α bir kapalı 1-form ise adı geçen manifoldlara lokal olarak konform Kaehler manifoldları denir. Bu çalışmada, çevreleyen uzay N bir lokal olarak konform Kaehler manifold ise holomorfik distribüsyon $\mathcal{D}$ nin integrallenebilmesi için koşullar elde edilmiştir.

1. INTRODUCTION

Let N be an n-dimensional almost Hermitian manifold with structure (J, g) and let M be a real m-dimensional manifold which is isometrically immersed in N. We have three typical classes of submanifolds. If T_xM is invariant by J for every $x \in M$, then it is said to be *holomorphic*, or alternatively *complex*, and in the case T_xM is anti-invariant by J it is called *a totally real submanifold.* These two classes of submanifolds have been investigated extensively from different viewpoints.

In 1978, the concept of *CR-submanifolds* was introduced by Aurel Bejancu [**1**] as a bridge between holomorphic and totally real submanifolds. Roughly speaking, their tangent bundle splits into a complex part of constant dimension and a totally real part, orthogonal to the first one. After its introduction, the definition was soon extended to other ambient spaces and gave rise to a large amount of literature which indicates it is an interesting subject in differential geometry.

Since locally conformal Kaehler manifolds are in main scope of this study, we shall mention about them. A Hermitian manifold whose metric is locally conformal to a Kaehler metric is called *a locally conformal Kaehler manifold.* For the sake of brevity we usually say *an l.c.K-manifold.* Its characterization has been given by Izu Vaisman [**16**] as follows :

> A Hermitian manifold M with the fundamental 2-form Ω is an l.c.K-manifold if and only if there exists on M a global closed 1-form α such that
>
> $$d\Omega = 2\alpha \wedge \Omega.$$

After that, Toyoko Kashiwada [**9**] gave the tensorial representation of l.c.K-manifolds which is a very efficient tool on account of our purposes.

In this thesis, we study about CR-submanifolds of locally conformal Kaehler manifolds. This thesis consists of five chapters:

Chapter 2 is devoted to remind basic definitions and notions which we will need in later discussions. Since, l.c.K-manifolds admits Hermitian metrics, hence Riemannian metrics, it is worthwhile to state the fundamental notions of Riemannian Geometry. The second section is reserved for submanifolds. In the third section, we introduce the notion of distribution which we will use to define the CR-submanifold of an almost Hermitian manifold. Also, the classical theorem of Frobenius is stated in this section. Eventually, in the last section we mention about f-structures which is introduced first in a paper of Kentaro Yano [**17**] dated 1963.

The third Chapter begins with the discussion of complex manifolds. In this first section, we collect many useful formulas that are needed hereafter as well as basic definitions. The second section, form the basis of this study not only because of the definition of CR-submanifolds, but also owing to the given related concepts. Moreover, in this section we present an important theorem of Blair and Chen [**4**] which is essential to justify the name "CR"-submanifold. And in the third section, we deal with the integrability of distributions on a CR-submanifold. This section is a preparation to the last section of the following chapter.

The fourth Chapter is reserved for l.c.K-manifolds. In the first section, we briefly introduce basic terms and then present two theorems both of which can be used to characterize l.c.K-manifolds. The second section is the place to use our prior knowledge and we do this by examining the integrability of CR-submanifolds of locally conformal Kaehler manifolds.

Finally, in Chapter 5 we give conclusion and recommendations.

2. ESSENTIAL MATTERS

2.1 Riemannian Manifolds

Let M be a real m-dimensional connected differentiable manifold of class C^∞ —throughout this study all the manifolds and tensor fields are assumed to be *differentiable* of class C^∞— covered by a system of coordinate neighborhoods $\{U; x^h\}$, where U denotes a neighborhood and x^h are local coordinates in U. Here the indices h, i, j run over the range $1, \ldots, m$. For any scalar field f and any vector field X on M, we define $Xf \in C^\infty(M)$ by

$$Xf = X^h \frac{\partial f}{\partial x^h}, \tag{2.1}$$

whereby X^h are the local components of X with respect to the natural frame $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\}$. Here and in the sequel we make use of *the Einstein convention*, that is, the repeated indices which appear once in superscript and once in subscript imply summation over their range.

A linear connection on M is defined as a mapping $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying the following conditions:

(i) $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z,$

(ii) $\nabla_X (fY + Z) = f\nabla_X Y + (Xf)Y + \nabla_X Z,$

whereby TM is the tangent bundle of M, $\mathfrak{X}(M)$ the module of differentiable sections of TM and $f \in C^\infty(M)$, X, Y, Z are vector fields on M. We say that $\nabla_X Y$ is *the covariant differentiation* of Y with respect to X. Also, we define the covariant differentiation of a function f with respect to X by

$$\nabla_X f = Xf. \tag{2.2}$$

Let S be a tensor field of type $(0, s)$ or $(1, s)$. *The covariant derivative* of S with respect to X is given by

$$(\nabla_X S)(X_1, \ldots, X_s) = \nabla_X (S(X_1, \ldots, X_s)) - \sum_{i=1}^{s} S(X_1, \ldots, \nabla_X X_i, \ldots, X_s) \tag{2.3}$$

for any vector field X_i. If the covariant derivative of S with respect to any vector field X on M is identically zero, that is,

$$\nabla_X S = 0 \tag{2.4}$$

for any $X \in \mathfrak{X}(M)$, then we say S is *parallel* with respect to ∇.

The Lie bracket of vector fields X and Y is defined by

$$[X,Y](f) = X(Yf) - Y(Xf) \tag{2.5}$$

for any scalar field f. The tensor field T of type $(1,2)$ defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{2.6}$$

for any vector fields X and Y on M is called *the torsion tensor* of the linear connection ∇ and if it is a vanishing tensor field then the connection ∇ is said to be *a torsion free connection.*

A Riemannian metric on a manifold M is a tensor field g of type $(0,2)$ satisfying the following conditions:

(i) g is symmetric, that is, $g(X,Y) = g(Y,X)$ for any vector fields X and Y on M,

(ii) g is positive definite, that is, $g(X,X) \geq 0$ for every vector field X and $g(X,X) = 0$ if and only if $X = 0$.

A manifold endowed with a Riemannian metric g is called *a Riemannian manifold.* When (ii) replaced by

(iii) g is nondegenerate, that is, $g(X,Y) = 0$ for every vector field Y on M implies $X = 0$,

g is called *a semi-Riemannian metric* and a manifold endowed with a semi-Riemannian metric is called *a semi-Riemannian manifold.* The length of a vector field X is denoted by $\|X\|$ and defined by means of metric tensor as

$$\|X\| = g(X,X). \tag{2.7}$$

The following well known theorem is *the miracle of semi-Riemannian geometry*:

Theorem 2.1. *Let (M, g) be a semi-Riemannian manifold. Then there exist a unique linear connection ∇ on M satisfying*

(i) $[X, Y] = \nabla_X Y - \nabla_Y X$, *that is, ∇ is torsion free,*

(ii) $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, *that is, ∇ is* metric

and it is characterized by the Koszul formula*:*

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned}$$

The connection mentioned in *Theorem 2.1* is called *the Levi-Civita connection.*

For a manifold M with a linear connection ∇, *the curvature tensor* R of type $(1, 3)$ is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.8}$$

whereby X, Y and Z vector fields on M. Now suppose g is a Riemannian metric on M and ∇ is the corresponding Levi-Civita connection. Then the covariant 4-tensor field defined by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W) \tag{2.9}$$

satisfies the following formulas

$$R(X, Y, Z, W) + R(Y, X, Z, W) = 0, \tag{2.10}$$

$$R(X, Y, Z, W) + R(X, Y, W, Z) = 0, \tag{2.11}$$

$$R(X, Y, Z, W) = R(Z, W, X, Y), \tag{2.12}$$

$$R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0 \tag{2.13}$$

for any vector fields X, Y, Z and W on M. Equations (2.10) to (2.13) are called *the symmetries* of the curvature tensor and the equation (2.13) has a special name: *first Bianchi identity.*

We define *the Ricci curvature tensor* as the trace of curvature tensor:

$$Ric(X, Y) = \sum_{i=1}^{m} g(R(E_i, X)Y, E_i), \tag{2.14}$$

whereby $\{E_1, \ldots, E_m\}$ stands for the local orthonormal frame on M. Notice that Ricci curvature tensor is a globally defined tensor field of type $(0, 2)$. The trace of the Ricci curvature tensor is called *the scalar curvature* of M and denoted by

$$r = \sum_{i=1}^{m} Ric(E_i, E_i). \tag{2.15}$$

Let $x \in M$ and let X and Y be two vectors of M at x which are orthonormal. Denote by γ the plane spanned by X and Y. Then *the sectional curvature* of this plane is denoted by $K(\gamma)$ and defined as follows:

$$K(\gamma) = g(R(X, Y)Y, X). \tag{2.16}$$

It can be verified that the sectional curvature of a plane is independent of the choice of plane's orthonormal basis. If $K(\gamma)$ is constant for all planes γ in the tangent space at x and for all points x in M, then M is called *a space of constant curvature* or *a real space form.*

Theorem 2.2 [10]. *Let M be a connected Riemannian manifold of dimension ≥ 3. If the sectional curvature $K(\gamma)$, where γ is a plane in T_xM, depends only on x, then M is a space of constant curvature.*

Let M be a real space form of constant sectional curvature c. The curvature tensor of M is given by

$$R(X, Y)Z = c\,\{g(Y, Z)X - g(X, Z)Y\} \tag{2.17}$$

for any vector fields X, Y and Z on M.

2.2 Submanifolds

Consider a mapping φ of a manifold M' into another manifold M''. The *differential* of φ at a point $x \in M'$ is a linear mapping $(\varphi_*)_x \colon T_xM' \to T_{\varphi(x)}M''$. Given some $X \in T_xM'$ and $f \in C^\infty(M'')$, $(\varphi_*)_x$ is defined by

$$(\varphi_*)_x(X)(f) = X(f \circ \varphi). \tag{2.18}$$

The rank of φ at a point $x \in M'$ is the dimension of $(\varphi_*)_x(T_xM')$. If it is equal to the dimension of M', then $(\varphi_*)_x$ is said to be *injective*. Moreover, if this is the case for every $x \in M'$, then φ is called *an immersion* and M' is said to be *a submanifold* of M''. An injective immersion is called *an imbedding.* Since our discussion is local, for a given submanifold, we may assume that it is an imbedded submanifold.

For a given open subset M' of a manifold M'' we may consider it as a submanifold of M'' in a natural manner. In this case M' is called *an open submanifold* of M''.

Now, let N be an n-dimensional Riemannian manifold endowed with Riemannian metric $\tilde{g}$ and let M be the m-dimensional submanifold of N. The metric g on M, defined by

$$g(X,Y) = \tilde{g}(X,Y) \tag{2.19}$$

for any vector fields X and Y on M, is called *the induced metric* on M. Notice that g is a Riemannian metric, and hence M is a Riemannian manifold with this induced metric g. Since the effects of both metrics g and $\tilde{g}$ are the same on TM, from now on, we denote both of them by g. Also, we denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and N, respectively .

Let ξ_x be a vector of N at a point x satisfying

$$g(X_x, \xi_x) = 0 \tag{2.20}$$

for any $X_x \in T_xM$. Then ξ_x is called *a normal vector* of M in N at x. We denote the vector bundle of all normal vectors of M in N, or in other words, the normal bundle of M in N, by $T^{\perp}M$. The restriction of the tangent bundle of N to M is the direct sum of TM and $T^{\perp}M$, that is,

$$TN|_M = TM \oplus T^{\perp}M. \tag{2.21}$$

Consider a vector field $\tilde{X}$ on N, which its restriction to TM is X. We call such $\tilde{X}$ as *an extension* of X. The subsequent propositions are essential:

Proposition 2.1 [5]. *Let X and Y be two vector fields on M and let $\tilde{X}$ and $\tilde{Y}$ be extensions of X and Y, respectively. Then $[\tilde{X},\tilde{Y}]|_M$ is independent of the extensions, and*

$$[\tilde{X},\tilde{Y}]|_M = [X,Y]. \tag{2.22}$$

Proposition 2.2 [**5**]. *Let X and Y be two vector fields on M and let $\tilde{X}$ and $\tilde{Y}$ be extensions of X and Y, respectively. Then $(\tilde{\nabla}_{\tilde{X}}\tilde{Y})|_M$ does not depend on the extensions. Denoting this by $\tilde{\nabla}_X Y$,*

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.23}$$

where ∇ is the Levi-Civita connection defined on the submanifold M with respect to g and $h(X, Y)$ is a normal vector field on M and is symmetric and bilinear in X and Y.

The formula (2.23) is called *the Gauss' formula*, we call the Levi-Civita connection ∇ *the induced connection* and h *the second fundamental form* of the submanifold M.

Given the vector fields X in TM and ξ in $T^{\perp}M$, we may decompose $\tilde{\nabla}_X\xi$ as

$$\tilde{\nabla}_X\xi = -A_\xi X + \nabla^{\perp}_X\xi, \tag{2.24}$$

whereby $-A_\xi X$ is the tangential component and $\nabla^{\perp}_X\xi$ is the normal component of $\tilde{\nabla}_X\xi$. We have the following Propositions:

Proposition 2.3 [**5**]. *$A_\xi X$ is bilinear in X and ξ and hence $A_\xi X$ at a point $x \in M$ depends only on X_x and ξ_x. Moreover, for each normal vector field ξ on M, we have*

$$g(A_\xi X, Y) = g(h(X, Y), \xi) \tag{2.25}$$

for any vector fields X and Y on M.

Proposition 2.4 [**5**]. *$\nabla^{\perp}$ is a metric connection in the normal bundle $T^{\perp}M$ of M in N with respect to the induced metric on $T^{\perp}M$.*

The formula (2.24) is called *the Weingarten's formula.* Also, we call the linear operator A_ξ *the shape operator* associated with ξ and the metric connection $\nabla^{\perp}$ the normal connection on M.

For the second fundamental form h, we define the covariant differentiation $\bar{\nabla}$ with respect to the connection in $(TM) \oplus (T^{\perp}M)$ by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{2.26}$$

for all vector fields X, Y and Z tangent to M.

Let R and $\tilde{R}$ be the curvature tensors on M and N, respectively. By a direct calculation, we obtain

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y \\ &\quad +(\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) \qquad (2.27)\end{aligned}$$

for every vector fields X, Y and Z on M.

By using equation (2.25) and (2.26), the equation (2.27) takes the form

$$\begin{aligned}g(\tilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + g(h(X,Z),h(Y,W)) \\ &\quad -g(h(Y,Z),h(X,W)), \qquad (2.28)\end{aligned}$$

whereby W is a vector field on M. The equation (2.28) is called *the equation of Gauss.*

Taking the normal components of $\tilde{R}(X,Y)Z$, denote by $\{\tilde{R}(X,Y)Z\}^{\perp}$, we get

$$\{\tilde{R}(X,Y)Z\}^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z). \qquad (2.29)$$

The equation (2.29) is called *the equation of Codazzi.*

We define the curvature tensor $R^{\perp}$ of the normal connection $\nabla^{\perp}$ on the normal bundle $T^{\perp}M$ by

$$R^{\perp}(X,Y)\xi = \nabla^{\perp}_X \nabla^{\perp}_Y \xi - \nabla^{\perp}_Y \nabla^{\perp}_X \xi - \nabla^{\perp}_{[X,Y]}\xi \qquad (2.30)$$

for any vector fields X, Y on M and a vector field ξ normal to M. Then, by taking a second vector field η normal to M, we have *the equation of Ricci*:

$$g(\tilde{R}(X,Y)\xi,\eta) = g(R^{\perp}(X,Y)\xi,\eta) - g([A_\xi, A_\eta]X,Y), \qquad (2.31)$$

whereby $[A_\xi, A_\eta] = A_\xi \circ A_\eta - A_\eta \circ A_\xi$.

Finally, we conclude this section by introducing some relevant notions. The reader may refer to **[5]** for a more comprehensive approach.

A submanifold M is said to be *totally geodesic* if the second fundamental form h vanishes identically, that is, $h(X,Y) = 0$ for any vector fields X and Y on M. For a unit normal vector field ξ in $T^{\perp}M$, if A_ξ is everywhere proportional to identity transformation I, then M is said to be *umbilical* with respect to ξ. If the submanifold M is umbilical with respect to every local normal section in M, then M is said to be *totally umbilical*.

Let $\xi_1, \ldots, \xi_{n-m}$ stands for an orthonormal basis of $T_x^{\perp}M$ and let $A^i = A_{\xi_i}$. Then *the mean curvature vector* H at a point $x \in M$ is defined by

$$H = \frac{1}{m}(\text{trace of } A^i)\xi_i \tag{2.32}$$

and is independent of the choice of the orthonormal basis. Here the index i run over the range $1, \ldots, n-m$. If the mean curvature vector H vanishes identically, then the submanifold M is called *a minimal submanifold*.

Proposition 2.5 [5]. *A totally umbilical submanifold is totally geodesic if and only if it is a minimal submanifold.*

2.3 Distributions

A p-dimensional distribution on an n-dimensional manifold N is a mapping $\mathcal{D}$ defined on N which assigns to each point x of N a p-dimensional linear subspace $\mathcal{D}_x$ of T_xN. That is,

$$\begin{aligned} \mathcal{D}:\ N &\longrightarrow TN \\ x &\longmapsto \mathcal{D}_x \subset T_xN. \end{aligned}$$

The distribution $\mathcal{D}$ is *differentiable* if for each x in N there is a neighborhood U of x and there are p differentiable vector fields $X_1, \ldots, X_p$ on U which span $\mathcal{D}$ at each point of U. A vector field X on N is said to belong to $\mathcal{D}$ if $X_x \in \mathcal{D}_x$ for every $x \in N$.

The distribution $\mathcal{D}$ is called *involutive* if $[X, Y]$ in $\mathcal{D}$ for any X and Y in $\mathcal{D}$. A submanifold M of N is called *an integral manifold* of the distribution $\mathcal{D}$ if $\varphi_*(T_xM) = \mathcal{D}_x$ for every $x \in M$, where φ_* is the differential of the imbedding φ of M into N. If there is no other integral manifold of $\mathcal{D}$ which contains M, then M is called *a maximal integral manifold* or *a leaf* of $\mathcal{D}$. The distribution $\mathcal{D}$ is said to be *completely integrable*, if, for every $x \in N$, there exists a unique integral manifold of $\mathcal{D}$ containing x.

The classical theorem of Frobenius can be stated as follows:

Theorem 2.3 [19]. *An involutive distribution $\mathcal{D}$ on N is integrable. Moreover, through every point $x \in N$ there passes a unique maximal integral manifold of $\mathcal{D}$ and every other integral manifold containing x is an open submanifold of this maximal one.*

Let ∇ be a linear connection on N. The distribution $\mathcal{D}$ is called *parallel* with respect to ∇ if we have $\nabla_X Y \in \mathcal{D}$ for all vector fields X on N and Y in $\mathcal{D}$.

Now, suppose N is endowed with two complementary distributions $\mathcal{D}$ and $\tilde{\mathcal{D}}$, that is, $\mathcal{D} \oplus \tilde{\mathcal{D}}$. Denote by P and Q the projections of TN to $\mathcal{D}$ and respectively to $\tilde{\mathcal{D}}$, and write

$$X = PX + QX, \tag{2.33}$$

whereby $PX \in \mathcal{D}$ and $QX \in \tilde{\mathcal{D}}$.

Theorem 2.4 [3]. *All the linear connections with respect to which both distributions $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are parallel, are given by*

$$\nabla_X Y = P\overset{\circ}{\nabla}_X PY + Q\overset{\circ}{\nabla}_X QY + PS(X, PY) + QS(X, QY) \tag{2.34}$$

for any vector fields X and Y on N, where $\overset{\circ}{\nabla}$ and S are, respectively, an arbitrary linear connection on N and an arbitrary tensor field of type $(1, 2)$ on N.

Proof. Let $\overset{\circ}{\nabla}$ be an arbitrary linear connection on N. Then any linear connection ∇ on N is given by

$$\nabla_X Y = \overset{\circ}{\nabla}_X Y + S(X, Y) \tag{2.35}$$

for any X, Y in TN, where S is an arbitrary tensor field of type $(1, 2)$ on N. Then $\nabla_X Y$ can be expressed by means of tensor fields P and Q as

$$\begin{aligned}
\nabla_X Y &= \nabla_X PY + \nabla_X QY \\
&= \overset{\circ}{\nabla}_X PY + S(X, PY) + \overset{\circ}{\nabla}_X QY + S(X, QY) \\
&= P\overset{\circ}{\nabla}_X PY + Q\overset{\circ}{\nabla}_X QY + PS(X, PY) + QS(X, QY) \\
&\quad + Q\overset{\circ}{\nabla}_X PY + QS(X, PY) + P\overset{\circ}{\nabla}_X QY + PS(X, QY) \\
&= P\overset{\circ}{\nabla}_X PY + Q\overset{\circ}{\nabla}_X QY + PS(X, PY) + QS(X, QY) \\
&\quad + Q\nabla_X PY + P\nabla_X QY.
\end{aligned} \tag{2.36}$$

The distributions $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are parallel with respect to ∇ if and only if $Q\nabla_X PY = 0$ and $P\nabla_X QY = 0$, identically. Therefore the equation (2.36) turns into the equation (2.34).

■

A tensor field $\tilde{F}$ of type $(1,1)$ is said to be *an almost product structure* on N if

$$\tilde{F}^2 X = X \tag{2.37}$$

for any X in TN. Now, define a tensor field F of type $(1,1)$ by

$$FX = PX - QX \tag{2.38}$$

for any vector field X on N. By a straightforward calculation, we obtain

$$\begin{aligned} F^2 X &= P(PX - QX) - Q(PX - QX) \\ &= PX + QX \\ &= X. \end{aligned}$$

The covariant derivative of F is defined by

$$(\nabla_X F)Y = \nabla_X FY - F(\nabla_X Y) \tag{2.39}$$

for all vector fields X, Y in TN. We say that the almost product structure F is *parallel* with respect to the linear connection ∇ if we have $\nabla_X F = 0$ for all $X \in TN$.

Theorem 2.5 [3]. *Both distributions $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are parallel with respect to ∇ if and only if the almost product structure F is parallel with respect to ∇.*

Proof. Suppose that $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are parallel with respect to ∇. Then we get

$$\begin{aligned} 0 &= 2Q\nabla_X PY - 2P\nabla_X QY \\ &\quad + P\nabla_X PY - P\nabla_X PY - Q\nabla_X QY + Q\nabla_X QY \\ &= \nabla_X PY - \nabla_X QY - P\nabla_X Y + Q\nabla_X Y \\ &= (\nabla_X F)Y. \end{aligned}$$

Conversely, assume $(\nabla_X F)Y = 0$. Taking into account (2.39) and (2.38), we have

$$\begin{aligned} 0 &= \nabla_X FY - F(\nabla_X Y) \\ &= \nabla_X PY - \nabla_X QY - P\nabla_X Y + Q\nabla_X Y \\ &= 2Q\nabla_X PY - 2P\nabla_X QY. \end{aligned}$$

But this is possible only if both $Q\nabla_X PY$ and $P\nabla_X QY$ are equal zero.

■

Now, let N be a Riemannian manifold with two complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ and ∇ be the Levi-Civita connection on N.

Theorem 2.6 [3]. *Both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel with respect to Levi-Civita connection ∇ if and only if they are integrable and their leaves are totally geodesic in N.*

Proof. Suppose that both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel with respect to ∇. Then

$$[X, Y] = \nabla_X Y - \nabla_Y X \in \mathcal{D} \tag{2.40}$$

for any vector fields X and Y in $\mathcal{D}$. This shows $\mathcal{D}$ is involutive and by the Theorem of Frobenius we conclude it is integrable.

Let M be a leaf of $\mathcal{D}$. For any vector fields X and Y on M, we have the Gauss' formula

$$h(X, Y) = \nabla_X Y - \nabla'_X Y,$$

whereby h is the second fundamental form of the immersion of M and ∇' denotes the Levi-Civita connection on M. It is obvious that $\nabla'_X Y$ tangent to M. Also, since $\nabla_X Y$ belongs to $\mathcal{D}$, it has no component in $T^\perp M$ too, which means the leaf of $\mathcal{D}$ is totally geodesic. A similar process can be carry out for $\mathcal{D}^\perp$.

Conversely, suppose $\mathcal{D}$ and $\mathcal{D}^\perp$ are integrable and their leaves are totally geodesic in N. Since $\nabla_X Y = \nabla'_X Y$ whenever X and Y tangent to M, $\nabla_X Y$ belongs to $\mathcal{D}$ whenever X and Y in $\mathcal{D}$. Similarly, $\nabla_U V$ belongs to $\mathcal{D}^\perp$ whenever U and V in $\mathcal{D}^\perp$. Now, we show that $\nabla_U Y$ belongs to $\mathcal{D}$ whenever U in $\mathcal{D}^\perp$ and Y in $\mathcal{D}$.

Since g is parallel with respect to ∇, for a vector field $V \in \mathcal{D}^{\perp}$, we find

$$\begin{aligned} 0 &= Ug(Y,V) \\ &= g(\nabla_U Y, V) + g(Y, \nabla_U V). \end{aligned}$$

Since $\nabla_U V$ belongs to $\mathcal{D}^{\perp}$, it follows

$$g(\nabla_U Y, V) = 0,$$

which implies $\nabla_U Y$ belongs to $\mathcal{D}$. Surely, a similar process can be applied to show $\mathcal{D}^{\perp}$ is parallel with respect to ∇.

■

From *Theorem 2.6* it follows that if N is endowed with two complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ that are parallel with respect to the Levi-Civita connection, then N is locally a Riemannian product $M \times M^{\perp}$, where M and $M^{\perp}$ are leaves of $\mathcal{D}$ and respectively $\mathcal{D}^{\perp}$.

2.4 f-structures

A non-null tensor field f of type $(1,1)$ on an m-dimensional connected manifold M is called *an f-structure* if it satisfies the relation

$$f^3 + f = 0. \tag{2.41}$$

We may decompose the unit tensor field I of type $(1,1)$ as

$$I = P + Q,$$

whereby $P = -f^2$ and $Q = f^2 + I$. It can be verified that the following relations

$$P^2 = P, \qquad Q^2 = Q \quad \text{and} \quad PQ = QP = 0 \tag{2.42}$$

hold. This means the operators P and Q applied to the tangent space at each point of the manifold are complementary projection operators. In other words, P and Q determine two distributions, say $\mathcal{D}$ and respectively $\tilde{\mathcal{D}}$, which are complementary. Moreover, the rank of f is constant, say p, requires that $\mathcal{D}$ is of dimension p and $\tilde{\mathcal{D}}$ is of dimension $m - p$ **[14]**.

For a real $2n$-dimensional differentiable manifold N, the tensor field J which is an endomorphism of T_xN at every point $x \in N$, is called *an almost complex structure* if $J^2 = -I$, whereby I is the identity transformation of T_xN.

Now, let X be an arbitrary vector field on M. Then

$$fPX = PfX = -f^3X = fX, \tag{2.43}$$

$$f^2PX = -f^4X = -f^3(fX) = f^2X = -PX, \tag{2.44}$$

$$fQX = f^3X + fX = 0, \tag{2.45}$$

$$f^2QX = f^4X + f^2X = -f^2X + f^2X = 0. \tag{2.46}$$

Equations (2.43) to (2.46) tell us f acts on $\mathcal{D}$ as an almost complex structure and on $\tilde{\mathcal{D}}$ as a null operator. Furthermore, if the rank of f is m then Q becomes a null tensor field which means f is an almost complex structure on M.

Eventually, define the tensor field N_f of type $(1, 2)$ by using f as

$$N_f(X, Y) = [fX, fY] + f^2[X, Y] - f[fX, Y] - f[X, fY] \tag{2.47}$$

for any vector fields X and Y on M. This tensor field is called *the Nijenhuis tensor* of f.

The distribution $\mathcal{D}$ is integrable if and only if $Q[PX, PY] = 0$ for any vector fields X and Y on N. Thus,

Proposition 2.7 [18]. *A necessary and sufficient condition for the distribution $\mathcal{D}$ to be integrable is that $QN_f(X, Y) = 0$, or $QN_f(PX, PY) = 0$, or $QN_f(fX, fY) = 0$ for any vector fields X and Y on N.*

The distribution $\tilde{\mathcal{D}}$ is integrable if and only if $P[QX, QY] = 0$ for any vector fields X and Y. Thus,

Proposition 2.8 [18]. *A necessary and sufficient condition for the distribution $\tilde{\mathcal{D}}$ to be integrable is that $N_f(QX, QY) = 0$ or equivalently $PN_f(QX, QY) = 0$ for any vector fields X and Y.*

3. CR-SUBMANIFOLDS

3.1 Complex Manifolds

Let N be a real $2n$-dimensional differentiable manifold. The tensor field J which is an endomorphism of T_xN at every point $x \in N$, is called *an almost complex structure* if $J^2 = -I$, whereby I denotes the identity transformation of T_xN. A manifold with a fixed almost complex structure is called *an almost complex manifold.*

Now, suppose N be an almost complex manifold with an almost complex structure J, then *the Nijenhuis tensor* of J is defined by

$$N_J(X,Y) = [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY] \tag{3.1}$$

for any vector fields X and Y on N. If N_J vanishes identically on N, then J is called *a complex structure* and N is said to be *a complex manifold* [**3**].

The term *Hermitian* is the analogue of the term Riemannian, in the complex case. Recall that a Riemannian manifold is a manifold with a Riemannian metric. Now, assume N is an almost complex manifold endowed with a Riemannian metric, say $\hat{g}$. If we derive a new metric g from $\hat{g}$ as

$$g(X,Y) = \hat{g}(X,Y) + \hat{g}(JX, JY),$$

then g is also a Riemannian metric and satisfies the relation

$$g(X,Y) = g(JX, JY). \tag{3.2}$$

A Riemannian metric satisfying equation (3.2) on an almost complex manifold is called *a Hermitian metric* and the manifold with the Hermitian metric is said to be *an almost Hermitian manifold.* Additionally, if the Nijenhuis tensor of J vanishes, it is called *a Hermitian manifold.*

Let N be an almost Hermitian manifold with structure (J, g), whereby J is an almost complex structure and g is a Hermitian metric. *The fundamental*

2-*form* Ω of N is defined by

$$\Omega(X,Y)=g(JX,Y) \tag{3.3}$$

for any vector fields X and Y on N. The exterior derivative of the fundamental 2-form Ω is given by

$$\begin{aligned} 3\,d\Omega(X,Y,Z) &= -X\Omega(Y,Z)-Y\Omega(Z,X)-Z\Omega(X,Y) \\ &\quad +\Omega([X,Y],Z)+\Omega([Y,Z],X)+\Omega([Z,X],Y) \end{aligned} \tag{3.4}$$

for any $X,Y,Z\in TN$. By using (3.3), the equation (3.4) takes the form

$$\begin{aligned} 3\,d\Omega(X,Y,Z) &= -Xg(JY,Z)-Yg(JZ,X)-Zg(JX,Y) \\ &\quad +g(J[X,Y],Z)+g(J[Y,Z],X)+g(J[Z,X],Y) \\ &= -g(\nabla_X JY,Z)+g(J\nabla_X Z,Y)-g(\nabla_Y JZ,X)+g(J\nabla_Y X,Z) \\ &\quad -g(\nabla_Z JX,Y)+g(J\nabla_Z Y,X) \\ &\quad -g(J\nabla_Y X,Z)+g(J\nabla_X Y,Z)-g(J\nabla_Z Y,X)+g(J\nabla_Y Z,X) \\ &\quad -g(J\nabla_X Z,Y)+g(J\nabla_Z X,Y) \\ &= -g(\nabla_X JY-J\nabla_X Y,Z)-g(\nabla_Y JZ-J\nabla_Y Z,X) \\ &\quad -g(\nabla_X JY-J\nabla_X Y,Z) \\ &= -g((\nabla_X J)Y,Z)-g((\nabla_Y J)Z,X)-g((\nabla_Z J)X,Y). \end{aligned} \tag{3.5}$$

As is seen the existence of Hermitian metric imposes no extra condition other than being a Riemannian manifold on an almost complex manifold. Now, we define a more restrictive class of Hermitian metrics: *A Kaehler metric* is a Hermitian metric satisfying $d\Omega=0$. An almost complex manifold endowed with a Kaehler metric is called *an almost Kaehler manifold*. Moreover, if the Nijenhuis tensor of the almost complex structure is identically zero, then it is called *a Kaehler manifold*.

Proposition 3.1 [11]. *For an almost Hermitian manifold N with structure (J,g) and Levi-Civita connection ∇, the following conditions are equivalent:*

(i) $\nabla_X J=0$

(ii) $N_J=0$ *and* $d\Omega=0$.

Hence, in an almost Hermitian manifold N if the almost complex structure J is parallel with respect to the Levi-Civita connection ∇, then it is a Kaehler manifold, and vice versa.

An almost Hermitian manifold N is called *a nearly Kaehler manifold* if

$$(\nabla_Z J)Z = 0 \tag{3.6}$$

for any vector field $Z \in TN$.
For any given vector fields X and Y on N, we have

$$\begin{aligned} 0 &= (\nabla_{X+Y} J)(X+Y) \\ &= (\nabla_{X+Y} J)X + (\nabla_{X+Y} J)Y \\ &= (\nabla_X J)X + (\nabla_Y J)X + (\nabla_X J)Y + (\nabla_Y J)Y \\ &= (\nabla_X J)Y + (\nabla_Y J)X, \end{aligned}$$

provided that N is nearly Kaehlerian.
Conversely, for an almost Hermitian manifold N if $(\nabla_X J)Y + (\nabla_Y J)X = 0$ holds for all vector fields X, Y tangent to N, then

$$\begin{aligned} 0 &= (\nabla_X J)Y + (\nabla_Y J)X \\ &= (\nabla_X J)Y + (\nabla_Y J)X + (\nabla_X J)X + (\nabla_Y J)Y \\ &= (\nabla_X J)(X+Y) + (\nabla_Y J)(X+Y) \\ &= (\nabla_{X+Y} J)(X+Y). \end{aligned}$$

Thus, the condition

$$(\nabla_X J)Y + (\nabla_Y J)X = 0 \tag{3.7}$$

is satisfied if and only if the condition (3.6) holds.

Proposition 3.2 [3]. *Let N be a nearly Kaehler manifold. Then the Nijenhuis tensor of J is given by*

$$N_J(X,Y) = 4J(\nabla_Y J)X \tag{3.8}$$

for any $X, Y \in TN$.

Proof. Since ∇ is a torsion free connection on N, we have

$$\begin{aligned} N_J(X,Y) &= [JX,JY]-[X,Y]-J[JX,Y]-J[X,JY] \\ &= \nabla_{JX}JY-\nabla_{JY}JX-\nabla_XY+\nabla_YX \\ &\quad +J\nabla_{JX}Y+J\nabla_YJX-J\nabla_XJY+J\nabla_{JY}X \\ &= (\nabla_{JX}JY-J\nabla_{JX}Y)-(\nabla_{JY}JX-J\nabla_{JY}X) \\ &\quad -J(\nabla_XJY-J\nabla_XY)+J(\nabla_YJX-J\nabla_YX) \\ &= (\nabla_{JX}J)Y-(\nabla_{JY}J)X-J((\nabla_XJ)Y)+J((\nabla_YJ)X). \end{aligned}$$

On the other hand, we know that N is a nearly Kaehler manifold. Using the equation (3.7), we get

$$\begin{aligned} N_J(X,Y) &= -(\nabla_YJ)JX+(\nabla_XJ)JY-2J((\nabla_XJ)Y) \\ &= (\nabla_YX+J\nabla_YJX)-(\nabla_XY+J\nabla_XJY)-2J((\nabla_XJ)Y \\ &= J(\nabla_YJX-J\nabla_YX)-J(\nabla_XJY-J\nabla_XY)-2J((\nabla_XJ)Y \\ &= J((\nabla_YJ)X)-3J((\nabla_XJ)Y) \\ &= -4J(\nabla_XJ)Y. \end{aligned}$$

■

We conclude this section by mentioning about sectional curvature of a Kaehler manifold.

Proposition 3.3 [11]. *The curvature tensor R of a Kaehler manifold N possess the following properties:*

(i) $R(JX,JY)=R(X,Y)$

(ii) $R(X,Y)\circ J=J\circ R(X,Y)$

for all vector fields X and Y on N.

Proposition 3.4 [11]. *Let V be a $2n$-dimensional real vector space with a complex structure J and R_1 and R_2 be two quadrilinear mappings $V\times V\times V\times V\to$* $\mathbf{R}$ *satisfying*

(i) $R(X,Y,Z,W)=-R(Y,X,Z,W)=-R(X,Y,W,Z)$

(ii) $R(X,Y,Z,W)=R(Z,W,X,Y)=0$

(iii) $R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W)=0$

(iv) $R(X,Y,Z,W)=R(JX,JY,Z,W)=R(X,Y,JZ,JW)$

for any X,Y,Z *and* W *in* V. *If* $R_1(X,JX,JX,X)=R_2(X,JX,JX,X)$ *for all* $X\in V$ *then* $R_1=R_2$.

Recall that for a Riemannian manifold M the sectional curvature $K(\gamma)$ of a plane γ in T_xM is defined by

$$K(\gamma)=g(R(X,Y)Y,X), \tag{3.9}$$

whereby X and Y is an orthonormal basis for γ. Now, let N be a Kaehler manifold with the complex structure J and γ denotes a plane in T_xN. If γ is invariant by the complex structure J, that is, $JX\in\gamma$ whenever $X\in\gamma$, then $K(\gamma)$ is called *the holomorphic sectional curvature* by γ. Suppose γ is invariant by J and X is a unit vector in γ then X and JX constitute an orthonormal basis for γ, which means

$$K(\gamma)=g(R(X,JX)JX,X). \tag{3.10}$$

Notice that *Proposition 3.4* implies the Riemannian curvature tensor R at x is determined by the holomorphic sectional curvatures $K(\gamma)$ of the planes γ which are invariant by complex structure J.

Theorem 3.1 [11]. *Let* N *be a connected Kaehler manifold of complex dimension* $n\geqslant 2$. *If the holomorphic sectional curvature* $K(\gamma)$, *where* γ *is a plane in* T_xN *invariant by complex structure* J, *depends only* x, *then* N *is a space of constant holomorphic sectional curvature, that is,* $K(\gamma)$ *is a constant for all planes* γ *in* T_xN *invariant by* J *and for all points* $x\in N$.

A Kaehler manifold of constant holomorphic sectional curvature is called *a complex space form.* In a complex space form N of constant holomorphic sectional curvature c, the curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X-g(X,Z)Y \\ &\quad +\Omega(Y,Z)JX-\Omega(X,Z)JY-2\Omega(X,Y)JZ\} \end{aligned} \tag{3.11}$$

for any vector fields X,Y and $Z\in TN$.

3.2 CR-Submanifolds of Almost Hermitian Manifolds

Let N be an n-dimensional almost Hermitian manifold with almost complex structure J and with a Hermitian metric g. Let M be a real m-dimensional Riemannian manifold which is isometrically immersed in N. Two important class of submanifolds are defined as follows:

M is called *a complex* or *an holomorphic submanifold* of N if

$$J(T_xM) = T_xM \tag{3.12}$$

for all $x \in M$, that is, if T_xM is invariant by J for all $x \in M$.

M is called *a totally real* or *an anti-invariant submanifold* of N if

$$J(T_xM) \subset T_x^{\perp}M \tag{3.13}$$

for all $x \in M$, that is, if T_xM is anti-invariant by J for all $x \in M$.

The concept of CR-submanifolds of an almost Hermitian manifold is situated between the above two classes of submanifolds. It is defined by Aurel Bejancu [**1**] as follows:

A real submanifold M of N is called *a CR-submanifold* if there exists a differentiable distribution $\mathcal{D}\colon x \to \mathcal{D}_x \subset T_xM$ on M satisfying

(i) $\mathcal{D}$ is holomorphic, that is, $J(\mathcal{D}_x) = \mathcal{D}_x$ for each $x \in M$

(ii) the complementary orthogonal distribution $\mathcal{D}^{\perp}\colon x \to \mathcal{D}_x^{\perp} \subset T_xM$ is anti-invariant, that is, $J(\mathcal{D}_x^{\perp}) \subset T_x^{\perp}M$.

We denote by p the complex dimension of the distribution $\mathcal{D}$ and by q the real dimension of the distribution $\mathcal{D}^{\perp}$. Notice that in the case of $q = 0$, the CR-submanifold M of N becomes a complex submanifold of N and in the case of $p = 0$, it is a totally real submanifold of N. If M is neither a complex submanifold nor a totally real submanifold, then it is called *a proper CR-submanifold* of N.

Consider any distribution $\mathcal{D}'$ on M which is invariant by J. Then for any X in $\mathcal{D}'$ and any U in $\mathcal{D}^{\perp}$ we have

$$g(X, U) = g(JX, JU) = 0$$

which implies X belongs to $\mathcal{D}$ and hence $\mathcal{D}$ is the maximal distribution invariant by J. Similarly, $\mathcal{D}^{\perp}$ is the maximal distribution anti-invariant by J.

We denote by symbols $\top$ and $\perp$ the tangential part and respectively the normal part of the corresponding vector or vector field.

For any vector field X tangent to M we can decompose JX as

$$JX = \phi X + \omega X, \tag{3.14}$$

whereby ϕX is the tangential part of JX, that is, $\phi X = \{JX\}^{\top}$ and ωX is the normal part of JX, that is, $\omega X = \{JX\}^{\perp}$. Then ϕ is an endomorphism of the tangent bundle TM of M and ω is a normal bundle valued 1-form on TM. Also, for any vector field ξ normal to M, we put

$$J\xi = B\xi + C\xi, \tag{3.15}$$

whereby $B\xi$ and $C\xi$ stand for the tangential part and the normal parts of $J\xi$, respectively, that is, $B\xi = \{J\xi\}^{\top}$ and $C\xi = \{J\xi\}^{\perp}$. Then B is a tangent bundle valued 1-form on $T^{\perp}M$ and C is an endomorphism of the normal bundle $T^{\perp}M$.

Theorem 3.2 [3]. *The submanifold M of N is a CR-submanifold if and only if*

(i) $\quad rank(\phi) = constant$

and

(ii) $\quad \omega \circ \phi = 0.$

Proof. Suppose that M is a CR-submanifold of an almost Hermitian manifold N with complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ which are invariant and anti-invariant by J, respectively. For any vector field X in TM, in view of (2.33), it follows

$$JX = JPX + JQX, \tag{3.16}$$

whereby JPX in TM and JQX in $T^{\perp}M$. Thus,

$$\phi X = JPX \qquad \text{and} \qquad \omega X = JQX. \tag{3.17}$$

Since for every X in TM we have JPX in $\mathcal{D}$, the rank of ϕ is constant, namely is equal $2p$. Moreover,

$$(\omega \circ \phi)X = \omega(JPX) = JQJPX$$

and since JPX in $\mathcal{D}$, $\omega \circ \phi = 0$, identically.

Conversely, assume that (i) and (ii) are satisfied. We define the distribution $\mathcal{D}$ by

$$\mathcal{D}_x = Im(\phi_x), \qquad \forall x \in M. \tag{3.18}$$

For a given vector field X tangent to M, ϕX belongs to $\mathcal{D}$. Moreover, we have

$$J\phi X = \phi^2 X + (\omega \circ \phi)X = \phi^2 X$$

which implies $\mathcal{D}$ is invariant by J.

Denote by $\mathcal{D}^{\perp}$ the complementary orthogonal distribution to $\mathcal{D}$ in TM. Then, for any U in $\mathcal{D}^{\perp}$ and Y in TM, we get

$$\begin{aligned} g(JU, Y) &= -g(U, JY) \\ &= -g(U, JPY) - g(U, JQY) \\ &= -g(U, JQY) \\ &= -g(U, \phi QY). \end{aligned}$$

Since the distribution $\mathcal{D}$ is defined as the image of ϕ, $g(JU, Y) = 0$. And this implies $\mathcal{D}^{\perp}$ is anti-invariant by J.

■

Theorem 3.3 [3]. *The submanifold M of N is a CR-submanifold if and only if*

(i) $rank(B) = constant$

and

(ii) $\phi \circ B = 0.$

Proof. Suppose that M is a CR-submanifold of an almost Hermitian manifold N with complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ which are invariant and anti-invariant by J, respectively. For any vector field X in $\mathcal{D}$ and ξ in $T^{\perp}M$, we have

$$g(B\xi, X) = g(B\xi, X) + g(C\xi, X) = g(B\xi + C\xi, X) = g(J\xi, X).$$

Also, we know that JX in $\mathcal{D}$ and since

$$g(J\xi, X) = -g(\xi, JX),$$

we have

$$g(B\xi, X) = 0$$

which implies $Im(B_x) \subset \mathcal{D}_x^{\perp}$ for each $x \in M$.

For any vector $U \in \mathcal{D}_x^{\perp}$, we have $JU \in T_x^{\perp}M$, and hence,

$$\begin{aligned} J(JU) &= BJU + CJU \\ J^2U &= BJU + CJU \\ -U &= BJU + CJU. \end{aligned}$$

But, we know $-U \in \mathcal{D}_x^{\perp}$, $BJU \in T_xM$ and $CJU \in T_x^{\perp}M$, and hence, we have $CJU = 0$ and $-U = BJU$, which implies $\mathcal{D}_x^{\perp} \subset Im(B_x)$. Thus, $\mathcal{D}^{\perp} = Im(B)$ shows that B is of constant rank.

Next, for each vector field ξ normal to M we have $B\xi \in \mathcal{D}^{\perp}$, and $JB\xi$ can be written as

$$JB\xi = \phi B\xi + \omega B\xi.$$

We know that $JB\xi$ normal to M, since $B\xi$ in $\mathcal{D}^{\perp}$, and also by definition $\omega B\xi$ in $T^{\perp}M$. Then, since $\phi B\xi$ tangent to M, it is identically zero.

Conversely, assume (i) and (ii) hold. We define the distribution $\mathcal{D}^{\perp}$ by

$$\mathcal{D}_x^{\perp} = Im(B_x), \qquad \forall x \in M. \tag{3.19}$$

Let ξ a be vector field normal to M. Then $B\xi$ belongs to $\mathcal{D}^{\perp}$. Moreover, for any vector field X on M, we have

$$g(JB\xi, X) = g(\phi B\xi, X) = 0,$$

which implies $\mathcal{D}^{\perp}$ is anti-invariant by J.

Let $\mathcal{D}$ be the complementary orthogonal distribution to $\mathcal{D}^{\perp}$ in TM. Then, for any vector field $X \in \mathcal{D}$ and $U \in \mathcal{D}^{\perp}$, we have

$$g(JX, U) = -g(X, JU) = 0.$$

Also, for any vector field ξ normal to M, since $B\xi \in \mathcal{D}^{\perp}$, we have

$$g(JX, \xi) = -g(X, J\xi) = -g(X, B\xi) = 0.$$

But this implies JX has component neither in $\mathcal{D}^{\perp}$ nor in $T^{\perp}M$ and hence $\mathcal{D}$ is invariant by J.

■

Proposition 3.5 [3]. *On each CR-submanifold M the vector bundle morphisms ϕ and C define f-structures on TM and $T^{\perp}M$, respectively.*

Proof. Suppose M is a CR-submanifold of the almost Hermitian manifold N. Then for any X in TM, we have

$$\begin{aligned} \phi X &= JPX, \\ \phi^2 X &= \phi JPX = \{J^2PX\}^{\top} = -PX, \\ \phi^3 X &= \phi(-PX) = \{-JPX\}^{\top} = -JPX = -\phi X, \end{aligned} \tag{3.20}$$

and hence,

$$\phi^3 X + \phi X = 0. \tag{3.21}$$

For any vector field ξ normal to M, we get

$$\begin{aligned} J\xi &= B\xi + C\xi \\ -\xi &= JB\xi + BC\xi + C^2\xi \\ 0 &= C^2\xi + \xi + JB\xi \\ 0 &= JC^2\xi + J\xi - B\xi \\ 0 &= JC^2\xi + C\xi \\ 0 &= C^3\xi + C\xi. \end{aligned} \tag{3.22}$$

■

Now, we introduce the notion of *CR-manifolds* [8]. Let M be a differentiable manifold and let $(TM)^{\mathbf{C}}$ be the *complexified* tangent bundle to M, that is, $(T_xM)^{\mathbf{C}} = T_xM \otimes_{\mathbf{R}} \mathbf{C}$ for every $x \in M$. *A CR-structure* on M is a complex subbundle $\mathcal{H}$ of $(TM)^{\mathbf{C}}$ such that $\mathcal{H}_x \cap \bar{\mathcal{H}}_x = 0$ and $\mathcal{H}$ is involutive. A manifold endowed with a CR-structure is called *a CR-manifold.*

The following theorem is essential in order to justify the name *CR-submanifold.*

Theorem 3.4 [4]. *Let M be a CR-submanifold of a* Hermitian *manifold N. Then M is a CR-manifold.*

Proof. Let $\mathcal{H}$ be the complex subbundle of $(TM)^{\mathbf{C}}$ defined by

$$\mathcal{H}_x = \{X - \sqrt{-1}\phi X \mid X \in \mathcal{D}_x\}. \tag{3.23}$$

Then, for any vector fields X and Y in $\mathcal{D}$, we have

$$\begin{aligned}
&[X - \sqrt{-1}\phi X, Y - \sqrt{-1}\phi Y] \\
&\qquad = [X, Y] - [X, \sqrt{-1}\phi Y] - [\sqrt{-1}\phi X, Y] + [\sqrt{-1}\phi X, \sqrt{-1}\phi Y] \\
&\qquad = [X, Y] - [\phi X, \phi Y] - \sqrt{-1}\{[X, \phi Y] + [\phi X, Y]\} \\
&\qquad = [X, Y] - [JX, JY] - \sqrt{-1}\{[X, JY] + [JX, Y]\}.
\end{aligned} \tag{3.24}$$

We will show that the equality (3.24) implies that $\mathcal{H}$ is involutive. To do so, we will use the fact that the Nijenhuis tensor of J vanishes.
Let X and Y be vector fields in $\mathcal{D}$. Then,

$$\begin{aligned}
0 &= N_J(JX, Y) \\
&= -[X, JY] - [JX, Y] + J([X, Y] - [JX, JY])
\end{aligned}$$

and by reordering terms, we get

$$[X, JY] + [JX, Y] = J([X, Y] - [JX, JY]).$$

Since $[X, JY] + [JX, Y]$ in TM, $[X, Y] - [JX, JY]$ has no component in $\mathcal{D}^\perp$. This implies $[X, Y] - [JX, JY]$ belongs to $\mathcal{D}$. That is,

$$[X, JY] + [JX, Y] = \phi([X, Y] - [JX, JY]). \tag{3.25}$$

Thus, by using (3.25) the equation (3.24) takes the form

$$\begin{aligned}
&[X - \sqrt{-1}\phi X, Y - \sqrt{-1}\phi Y] \\
&\qquad = [X, Y] - [JX, JY] - \sqrt{-1}\{\phi([X, Y] - [JX, JY])\},
\end{aligned} \tag{3.26}$$

which implies $\mathcal{H}$ is involutive.

■

3.3 Integrability of Distributions on a CR-Submanifold

In this section, we investigate the integrability of the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on a CR-submanifold M of an almost Hermitian manifold N.

Theorem 3.5 [2]. *Let M be a CR-submanifold of an almost Hermitian manifold N. Then the distribution $\mathcal{D}$ is integrable if and only if*

$$\{N_J(X,Y)\}^\top = N_\phi(X,Y) \tag{3.27}$$

for any vector fields X and Y in $\mathcal{D}$.

Proof. Since X and Y belong to $\mathcal{D}$, the Nijenhuis tensor of J

$$N_J(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY]$$

becomes

$$\begin{aligned} N_J(X,Y) &= [\phi X,\phi Y] - [X,Y] - J[\phi X,Y] - J[X,\phi Y] \\ &= [\phi X,\phi Y] - P[X,Y] - Q[X,Y] - \phi[\phi X,Y] - \omega[\phi X,Y] \\ &\qquad - \phi[X,\phi Y] - \omega[X,\phi Y]. \end{aligned}$$

By rearranging terms and using (3.20), we get

$$\begin{aligned} N_J(X,Y) &= [\phi X,\phi Y] + \phi^2[X,Y] - \phi[\phi X,Y] - \phi[X,\phi Y] \\ &\qquad - Q[X,Y] - \omega[\phi X,Y] - \omega[X,\phi Y] \\ &= N_\phi(X,Y) - Q[X,Y] - \omega([\phi X,Y] + [X,\phi Y]). \end{aligned} \tag{3.28}$$

Now, if we take the tangential part of both sides, we have

$$\{N_J(X,Y)\}^\top = N_\phi(X,Y) - Q[X,Y] \tag{3.29}$$

which implies the condition (3.37) is equivalent to the condition $Q[X,Y] = 0$. Since the vector fields X and Y in $\mathcal{D}$, $Q[X,Y] = 0$ is the necessary and sufficient condition for $\mathcal{D}$ to be integrable.

■

Theorem 3.6 [2]. *Let M be a CR-submanifold of an almost Hermitian manifold N. Then the distribution $\mathcal{D}$ is integrable if and only if*

$$\{N_J(X,Y)\}^\perp = 0 \qquad \text{and} \qquad QN_\phi(X,Y) = 0 \tag{3.30}$$

for any vector fields X and Y in $\mathcal{D}$.

Proof. If we take the normal part of both sides of the equality (3.28), we get

$$\{N_J(X,Y)\}^\perp = -\omega([\phi X, Y] + [X, \phi Y]). \tag{3.31}$$

Suppose that $\mathcal{D}$ is integrable. Since $X, Y \in \mathcal{D}$, $[\phi X, Y]$ and $[X, \phi Y]$ belong to $\mathcal{D}$. Thus, $\{N_J(X,Y)\}^\perp = 0$. Also, since $\mathcal{D}$ is integrable, $N_\phi(X,Y)$ has no component in $\mathcal{D}^\perp$ for any vector fields X and Y in $\mathcal{D}$, that is, $QN_\phi(X,Y) = 0$ for any vector fields X and Y in $\mathcal{D}$.

Conversely, suppose (3.30) is satisfied for any vector fields X and Y in $\mathcal{D}$. Since $\mathcal{D}$ is invariant by J, $\{N_J(JX,Y)\}^\perp = 0$ for any vector fields $X, Y \in \mathcal{D}$. Thus,

$$\begin{aligned} 0 &= \{N_J(JX,Y)\}^\perp \\ &= -\omega([\phi JX, Y] + [JX, \phi Y]) \\ &= -\omega(-[X,Y] + [JX, JY]). \end{aligned}$$

And by using (3.18), we conclude

$$Q([JX, JY] - [X,Y]) = 0. \tag{3.32}$$

Also, if we project the each term of the equation (3.1) to $\mathcal{D}^\perp$, we obtain

$$QN_J(X,Y) = Q([JX, JY] - [X,Y]) - \omega([JX, Y] + [X, JY]).$$

Since $\omega([JX,Y] + [X,JY]) \in T^\perp M$, $QN_J(X,Y) = Q([JX,JY] - [X,Y])$ and from (3.32), it follows

$$QN_J(X,Y) = 0. \tag{3.33}$$

Now, if we combine (3.31) and (3.30), we get

$$\begin{aligned} 0 &= \{N_J(X,Y)\}^\perp \\ &= -\omega([\phi X, Y] + [X, \phi Y]), \end{aligned} \tag{3.34}$$

and by substituting (3.34) into (3.28), we get

$$\begin{aligned} N_J(X,Y) &= N_\phi(X,Y) - Q[X,Y] \\ QN_J(X,Y) &= QN_\phi(X,Y) - Q[X,Y] \\ QN_J(X,Y) &= -Q[X,Y]. \end{aligned}$$

Therefore, taking into account (3.33), $\mathcal{D}$ is integrable.

∎

Consider the case N is a Hermitian manifold. Additionally, if we have $N_\phi(X,Y)=0$ then the equation (3.29) becomes

$$Q[X,Y]=0$$

for any vector fields X and Y in $\mathcal{D}$. Thus, we have the following Corollary:

Corollary 3.1 [3]. *Let M be a CR-submanifold of a Hermitian manifold N. The distribution $\mathcal{D}$ is integrable if and only if the Nijenhuis tensor of ϕ vanishes identically on $\mathcal{D}$.*

Theorem 3.7 [3]. *Let M be a CR-submanifold of an almost Hermitian manifold N. The distribution $\mathcal{D}^\perp$ is integrable if and only if the Nijenhuis tensor of ϕ vanishes identically on $\mathcal{D}^\perp$.*

Proof. For any vector fields U and V in $\mathcal{D}^\perp$, the Nijenhuis tensor of ϕ

$$N_\phi(U,V)=[\phi U,\phi V]-P[U,V]-\phi([\phi U,V]+[U,\phi V]) \tag{3.35}$$

becomes

$$N_\phi(U,V)=-P[U,V]$$

which means if the Nijenhuis tensor of ϕ vanishes identically on $\mathcal{D}^\perp$, then $P[U,V]=0$ for any vector fields U and V in $\mathcal{D}^\perp$.

∎

Theorem 3.8 [13]. *Let M be a CR-submanifold of a nearly Kaehler manifold N. Then the distribution $\mathcal{D}$ is integrable if and only if the following conditions are satisfied:*

$$h(X,JY)=h(JX,Y) \tag{3.36}$$

and

$$N_J(X,Y)\in\mathcal{D} \tag{3.37}$$

for any $X,Y\in\mathcal{D}$.

Proof. Since the ambient manifold N is a nearly Kaehler manifold, in virtue of (3.7) the following equation

$$\begin{aligned}
[JX,Y]+[X,JY] &= \tilde{\nabla}_{JX}Y-\tilde{\nabla}_{Y}JX+\tilde{\nabla}_{X}JY-\tilde{\nabla}_{JY}X \\
&= -((\tilde{\nabla}_{Y}J)X+J\tilde{\nabla}_{Y}X)+((\tilde{\nabla}_{X}J)Y+J\tilde{\nabla}_{X}Y) \\
&\quad +\tilde{\nabla}_{JX}Y-\tilde{\nabla}_{JY}X \\
&= -2(\tilde{\nabla}_{Y}J)X+[(\tilde{\nabla}_{Y}J)X+(\tilde{\nabla}_{X}J)Y] \\
&\quad +J(\tilde{\nabla}_{X}Y-\tilde{\nabla}_{Y}X)+\tilde{\nabla}_{JX}Y-\tilde{\nabla}_{JY}X
\end{aligned} \tag{3.38}$$

becomes

$$[JX,Y]+[X,JY]=-2(\tilde{\nabla}_{Y}J)X+J[X,Y]+\tilde{\nabla}_{JX}Y-\tilde{\nabla}_{JY}X, \tag{3.39}$$

whereby X and Y are vector fields in $\mathcal{D}$. Also, by using (3.8) it follows

$$[JX,Y]+[X,JY]=\frac{1}{2}J(N_J(X,Y))+J[X,Y]+\tilde{\nabla}_{JX}Y-\tilde{\nabla}_{JY}X. \tag{3.40}$$

Since JX and JY are in $\mathcal{D}\subset TM$, we can use (2.23). Then (3.40) takes the form

$$\begin{aligned}
[JX,Y]+[X,JY] &= \frac{1}{2}J(N_J(X,Y))+J[X,Y]+\nabla_{JX}Y-\nabla_{JY}X \\
&\quad +h(JX,Y)-h(X,JY).
\end{aligned} \tag{3.41}$$

And by reordering terms in the above equation, we get

$$\begin{aligned}
h(X,JY)-h(JX,Y) &= \frac{1}{2}J(N_J(X,Y))+J[X,Y] \\
&\quad +\nabla_{JX}Y-\nabla_{JY}X-[JX,Y]-[X,JY] \\
&= \frac{1}{2}J(N_J(X,Y))+J[X,Y] \\
&\quad +\nabla_{Y}JX-\nabla_{X}JY.
\end{aligned} \tag{3.42}$$

Now, suppose $\mathcal{D}$ is integrable. Then, since $J[X,Y]\in\mathcal{D}$, (3.42) becomes

$$h(X,JY)-h(JX,Y)=\frac{1}{2}J(N_J(X,Y)). \tag{3.43}$$

Also, by combining (3.27) and (3.30) we have

$$h(X,JY)-h(JX,Y)=\frac{1}{2}J(N_\phi(X,Y)) \tag{3.44}$$

and since $N_\phi(X,Y)$ has no component in $\mathcal{D}^\perp$, (3.36) and (3.37) hold.

Conversely, assume (3.36) and (3.37) are satisfied. Substituting (3.36) into (3.42), we get

$$J[X,Y] = \nabla_X JY - \nabla_Y JX - \frac{1}{2} J(N_J(X,Y)). \tag{3.45}$$

Since $\nabla_X JY$ and $\nabla_Y JX$ tangent to M, $J[X,Y] \in TM$. Now, let Z be a vector field in $\mathcal{D}^{\perp}$, then

$$g(J[X,Y], JZ) = 0 \tag{3.46}$$

and using (3.2) we obtain

$$g([X,Y], Z) = 0, \tag{3.47}$$

which means that $\mathcal{D}$ is integrable.

■

Theorem 3.9 [15]. *Let M be a CR-submanifold of a nearly Kaehler manifold N. Then the distribution $\mathcal{D}$ is integrable if and only if*

$$(\tilde{\nabla}_X J)Y \in \mathcal{D} \tag{3.48}$$

and (3.36) *for any $X, Y \in \mathcal{D}$.*

Proof. The proof follows from (3.8) and *Theorem 3.8*.

■

Corollary 3.2 [3]. *Let M be a CR-submanifold of a nearly Kaehler manifold N. Then the distribution $\mathcal{D}$ is integrable if and only if* (3.36) *is satisfied and*

$$\{N_J(X,U)\}^{\top} \in \mathcal{D}^{\perp} \tag{3.49}$$

for any $X \in \mathcal{D}$ and $U \in \mathcal{D}^{\perp}$.

Proof. For any vector fields X, Y, Z tangent to N, in virtue of (3.2), we have

$$Xg(JY, Z) = -Xg(Y, JZ) \tag{3.50}$$

and since $\tilde{\nabla}$ is a metric connection, we get

$$g(\tilde{\nabla}_X JY, Z) + g(JY, \tilde{\nabla}_X Z) = -g(\tilde{\nabla}_X Y, JZ) - g(Y, \tilde{\nabla}_X JZ) \tag{3.51}$$

and by reordering terms, the equation (3.51) turns into

$$\begin{aligned} g(\tilde{\nabla}_X JY - J\tilde{\nabla}_X Y, Z) &= -g(\tilde{\nabla}_X JZ - J\tilde{\nabla}_X Z, Y) \\ g((\tilde{\nabla}_X J)Y, Z) &= -g((\tilde{\nabla}_X J)Z, Y). \end{aligned} \tag{3.52}$$

Now, let $X, Y \in \mathcal{D}$ and $U \in \mathcal{D}^{\perp}$. Suppose (3.48) holds. Then by using (3.52)

$$-g((\tilde{\nabla}_X J)U, Y) = g((\tilde{\nabla}_X J)Y, U) = 0 \tag{3.53}$$

and, by using (3.2), we get

$$-g(J(\tilde{\nabla}_X J)U, JY) = 0, \tag{3.54}$$

which means that $J(\tilde{\nabla}_X J)U$ has no component in $\mathcal{D}$ and, in virtue of (3.8), we conclude that the tangential part of $N_J(X, U)$ must be in $\mathcal{D}^{\perp}$. Since all this steps are reversible, the equation (3.48) holds if and only if (3.49) is satisfied.

■

We denote by ν the complementary orthogonal subbundle to $J\mathcal{D}^{\perp}$ in $T^{\perp}M$. Hence, we have

$$T^{\perp}M = J\mathcal{D}^{\perp} \oplus \nu, \qquad J\mathcal{D}^{\perp} \perp \nu. \tag{3.55}$$

Let $x \in M$ and $\zeta \in \nu_x$, $X \in \mathcal{D}_x$, $U \in \mathcal{D}_x^{\perp}$. Then we get

$$g(J\zeta, X) = -g(\zeta, JX) = 0, \tag{3.56}$$

$$g(J\zeta, U) = -g(J\zeta, JU) = 0, \tag{3.57}$$

$$g(J\zeta, JU) = g(\zeta, U) = 0. \tag{3.58}$$

From (3.56), (3.57) and (3.58) we deduce ν is invariant by J, that is,

$$J(\nu_x) = \nu_x \qquad \text{for each} \quad x \in M. \tag{3.59}$$

Proposition 3.6 [3]. *The condition* (3.36) *is satisfied if and only if*

$$g(h(X, JY) - h(Y, JX), JU) = 0 \tag{3.60}$$

for any $X, Y \in \mathcal{D}$ and $U \in \mathcal{D}^{\perp}$.

Proof. In virtue of (3.42), we have

$$\begin{aligned} g(h(X, JY) - h(Y, JX), \zeta) &= \frac{1}{2} g(JN_J(X, Y), \zeta) + g(J[X, Y], \zeta) \\ &= -\frac{1}{2} g(N_J(X, Y), J\zeta) \\ &= -\frac{1}{2} g(\{N_J(X, Y)\}^{\perp}, J\zeta), \end{aligned} \tag{3.61}$$

whereby ζ is a vector field in ν. Also, by substituting (3.31) into (3.61), we get

$$\begin{aligned} g(h(X,JY)-h(Y,JX),\zeta) &= \frac{1}{2}g(\omega([\phi X,Y]+[X,\phi Y]),J\zeta) \\ &= \frac{1}{2}g(J([JX,Y]+[X,JY]),J\zeta) \\ &= \frac{1}{2}g(([JX,Y]+[X,JY]),\zeta) \\ &= 0, \end{aligned}$$

which implies the projection of $h(X,JY)-h(Y,JX)$ onto ν is identically zero. We also know $h(X,JY)-h(Y,JX)$ has no component in TM which means if (3.60) is satisfied then $h(X,JY)-h(Y,JX)$ is equal to zero.

■

Theorem 3.10 [15]. *Let M be a CR-submanifold of a nearly Kaehlerian manifold N. Then the distribution $\mathcal{D}^\perp$ is integrable if and only if*

$$g((\tilde{\nabla}_U J)V,X)=0 \tag{3.62}$$

for any $U,V\in\mathcal{D}^\perp$ and $X\in\mathcal{D}$.

Proof. First, by using (3.7) and (3.52), (3.5) can be rewritten as

$$d\Omega(U,V,X)=-g((\tilde{\nabla}_U J)V,X) \tag{3.63}$$

for any $U,V\in\mathcal{D}^\perp$ and $X\in\mathcal{D}$. Also, by using (3.3) and (3.4), the equation (3.5) takes the form

$$\begin{aligned} 3\,d\Omega(U,V,X) &= Ug(V,JX)-Vg(JX,U)-Xg(JU,V) \\ &\quad -g([U,V],JX)-g([V,X],JU)-g([X,U],JV) \\ &= -g([U,V],JX). \end{aligned} \tag{3.64}$$

We know that $\mathcal{D}^\perp$ is integrable if and only if $[U,V]\in\mathcal{D}^\perp$ for any $U,V\in\mathcal{D}^\perp$. Thus, in virtue of (3.63) and (3.64), we conclude that (3.62) is the necessary and sufficient condition for the integrability of $\mathcal{D}^\perp$.

■

Corollary 3.3 [13]. *Let M be a CR-submanifold of a nearly Kaehler manifold N. The distribution $\mathcal{D}^\perp$ is integrable if and only if*

$$g(h(U,X),JV)=g(h(V,X),JU) \tag{3.65}$$

for any $U,V\in\mathcal{D}^\perp$ and $X\in\mathcal{D}$.

Proof. Suppose that $\mathcal{D}^\perp$ is integrable. Taking into account (2.24) and (2.25), and using *Theorem 3.10*, we have

$$\begin{aligned} 0 &= g((\tilde{\nabla}_U J)V - (\tilde{\nabla}_V J)U, X) \\ &= g(\tilde{\nabla}_U JV - \tilde{\nabla}_V JU, X) + g([U,V], JX) \\ &= g(-A_{JV}U + A_{JU}V, X), \end{aligned} \tag{3.66}$$

and hence (3.65) is satisfied.

Conversely, assume that (3.65) fulfills. Then we have

$$g((\tilde{\nabla}_U J)V - (\tilde{\nabla}_V J)U, X) - g([U,V], JX) = 0 \tag{3.67}$$

and using (3.7), we get

$$g((\tilde{\nabla}_U J)V, X) = 0. \tag{3.68}$$

■

Proposition 3.7 [3]. *Let M be a CR-submanifold of a nearly Kaehler manifold N. If $\mathcal{D}^\perp$ is integrable then each leaf of $\mathcal{D}^\perp$ is immersed in M as a totally geodesic submanifold if and only if*

$$g(h(U,X), JV) = 0 \tag{3.69}$$

for any $U, V \in \mathcal{D}^\perp$ and $X \in \mathcal{D}$.

Proof. We have to show that $\nabla_U V \in \mathcal{D}^\perp$ if and only if (3.69) is satisfied. Since $\mathcal{D}^\perp$ is integrable, by virtue of (3.62) and (2.25), we have

$$\begin{aligned} g(\nabla_U V, JX) &= g(\tilde{\nabla}_U V, JX) = -g(\tilde{\nabla}_U JV, X) \\ &= g(A_{JV}U, X) = g(h(U,X), JV). \end{aligned}$$

■

4. LOCALLY CONFORMAL KAEHLER MANIFOLDS

4.1 Locally Conformal Kaehler Manifolds

We first mention about the conformal changes of the metric in the Riemannian case and present some corresponding formulas [**5**]. Afterwards we give the definition of a locally conformal Kaehler manifold.

Let N be an n-dimensional Riemannian manifold with metric tensor g and $\sigma\colon N \to \mathbf{R}$ be a C^∞ function. Then

$$g^* = e^{-2\sigma} g \tag{4.1}$$

defines a new metric tensor on N which does not change the angle between two vectors at a point. Hence it is a conformal change of the metric. In particular, if the function σ is a constant, the conformal transformation is said to be *homothetic*.

Let ∇^* denote the covariant differentiation with respect to g^*. Then, we have

$$\nabla^*_X Y = \nabla_X Y - \alpha(X)Y - \alpha(Y)X + g(X,Y)\alpha^\sharp \tag{4.2}$$

for any vector fields X, Y, whereby α is a 1-form given by

$$\alpha = d\sigma \tag{4.3}$$

and $\alpha^\sharp$ is the dual vector field of α, that is,

$$g(\alpha^\sharp, X) = \alpha(X). \tag{4.4}$$

Let R^* denote the curvature tensor of the Riemannian metric g^* and put

$$\begin{aligned} s(X,Y) &= -(\nabla_X \alpha)Y - \alpha(X)\alpha(Y) + \frac{1}{2}\|\alpha^\sharp\| g(X,Y), && (4.5)\\ g(SX,Y) &= s(X,Y). && (4.6) \end{aligned}$$

Then, we have

$$\begin{aligned} R^*(X,Y)Z &= R(X,Y)Z - s(Y,Z)X + s(X,Z)Y \\ &\quad - g(Y,Z)SX + g(X,Z)SY \end{aligned} \tag{4.7}$$

for any vector fields X, Y and Z.

Now, define the $(1,3)$ tensor field C by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - L(Y,Z)X + L(X,Z)Y \\ &\quad + g(Y,Z)NX + g(X,Z)NY \end{aligned} \tag{4.8}$$

for any vector fields X, Y and Z, whereby

$$L(X,Y) = -\frac{1}{n-2}Ric(X,Y) + \frac{r}{2(n-1)(n-2)}g(X,Y), \tag{4.9}$$

$$g(NX,Y) = L(X,Y). \tag{4.10}$$

Notice that, here r denotes the scalar curvature of the manifold N. Let C^* be the corresponding tensor field of type $(1,3)$ associated with g^*. It can be shown that

$$C = C^*, \tag{4.11}$$

that is, C is invariant under any conformal change of the metric. We call this tensor field *the Weyl conformal curvature tensor*, or simply *conformal curvature tensor*.

Let N be a real $2n$-dimensional Hermitian manifold with the structure (J,g), whereby J is an almost complex structure and g is a Hermitian metric. Then N is *a locally conformal Kaehler manifold (an l.c.K-manifold)* if there is an open cover $\{U_i\}_{i\in I}$ of N and a family $\{\sigma_i\}_{i\in I}$ of C^∞ functions $\sigma_i\colon U_i \to \mathbf{R}$ so that each local metric

$$g_i = e^{-2\sigma_i} g|_{U_i} \tag{4.12}$$

is a Kaehler metric. Also, N is *a globally conformal Kaehler manifold (a g.c.K-manifold)* if there is a C^∞ function $\sigma\colon N \to \mathbf{R}$ so that the metric $e^{-2\sigma}g$ is a Kaehler metric. Let Ω and Ω_i be the fundamental 2-forms associated with (J,g) and (J,g_i), respectively. Then, by using (3.3), (4.12) yields

$$\Omega_i = e^{-2\sigma_i}\Omega|_{U_i}. \tag{4.13}$$

Theorem 4.1 [7]. *The Hermitian manifold (N,J,g) is an l.c.K-manifold if and only if there exists a globally defined closed 1-form α on N so that*

$$d\Omega = 2\alpha \wedge \Omega. \tag{4.14}$$

Proof. Suppose N is an l.c.K-manifold. Let us take the exterior derivative of both sides of (4.13):

$$d\Omega_i = d(e^{-2\sigma_i}) \wedge \Omega + e^{-2\sigma_i} d\Omega. \tag{4.15}$$

Since Ω_i is associated with a Kaehler metric g_i, then (4.15) gives

$$d\Omega = 2d\sigma_i \wedge \Omega \tag{4.16}$$

on U_i. Hence, we have

$$d(\sigma_i - \sigma_j) \wedge \Omega = 0 \tag{4.17}$$

on the intersection of U_i and U_j. Therefore as Ω is nondegenerate $d\sigma_i = d\sigma_j$ on $U_i \cap U_j$ so that the local 1-forms $d\sigma_i$ glue up to a globally defined closed, say $d(d\sigma_i) = 0$, 1-form α on N, that is,

$$\alpha|_{U_i} = d\sigma_i. \tag{4.18}$$

Conversely, assume that there exist a globally defined closed 1-form α satisfying (4.14). By the classical *Poincaré lemma* there is an open cover $\{U_i\}_{i\in I}$ of N and a family of C^∞ functions $\sigma_i \colon U_i \to \mathbf{R}$ such that $\alpha = d\sigma_i$ on U_i. Since on U_i we have

$$d\Omega - (2d\sigma_i \wedge \Omega) = 0$$

by multiplying the above equation with $e^{-2\sigma_i}$, we get

$$e^{-2\sigma_i} d\Omega \; + \; d(e^{-2\sigma_i}) \wedge \Omega = d(e^{-2\sigma_i}\Omega) = 0, \tag{4.19}$$

which implies $e^{-2\sigma_i} g$ is a Kaehler metric on U_i.

■

We call the closed 1-form α satisfying (4.14) *the Lee form* of the l.c.K-manifold N. The following proposition is also useful to characterize l.c.K-manifolds:

Proposition 4.1 [9]. *A Hermitian manifold* (N, J, g) *is l.c.K if and only if there exists a global closed* 1*-form* α *satisfying*

$$\begin{aligned}(\nabla_X \Omega)(Y,Z) &= \alpha(Z)\Omega(X,Y) - \alpha(Y)\Omega(X,Z) \\ &\quad + \beta(Z)g(X,Y) - \beta(Y)g(X,Z)\end{aligned} \tag{4.20}$$

for any vector fields X, Y *and* Z *tangent to* N*, where* ∇ *denotes the covariant differentiation with respect to* g *and the* 1*-form* β *is defined by*

$$\beta(X) = -\alpha(JX) \tag{4.21}$$

for any vector field X tangent to N.

Proof. Suppose (N, J, g) be an l.c.K-manifold. From the proof of *Theorem 4.1* we know that local 1-forms $\{d\sigma_i\}_{i\in I}$ glue up to a globally defined closed 1-form α. Let ∇^i denotes the covariant differentiation with respect to g_i on U_i. Since each g_i and g are conformally related, $\{\nabla^i\}_{i\in I}$ glue up to a globally defined connection, say D, and in virtue of (4.2), we get

$$D_X Y = \nabla_X Y - \alpha(X)Y - \alpha(Y)X + g(X,Y)\alpha^\sharp \tag{4.22}$$

for any vector fields X, Y, Z, whereby $\alpha^\sharp$ is the vector field given by $g(\alpha^\sharp, X) = \alpha(X)$. Since each g_i for $i \in I$ is a Kaehler metric, we have $(D_X J)Y = 0$ which yields

$$\begin{aligned} D_X(JY) &= J(D_X Y), \\ g(D_X(JY), Z) &= g(J(D_X Y), Z), \\ g(D_X(JY), Z) &= -g(D_X Y, JZ). \end{aligned} \tag{4.23}$$

Substituting (4.22) into (4.23), we obtain

$$\begin{aligned} g(\nabla_X JY, Z) + g(\nabla_X Y, JZ) &= \alpha(X)g(JY, Z) + \alpha(X)g(Y, JZ) \\ &\quad +\alpha(JY)g(X, Z) + \alpha(Y)g(X, JZ) \\ &\quad -\alpha(Z)g(X, JY) - \alpha(JZ)g(X, Y). \end{aligned} \tag{4.24}$$

By canceling out the first two terms on the left hand-side of (4.24) and using (3.3) and (4.21), the above equation becomes

$$\begin{aligned} g((\nabla_X J)Y, Z) &= \alpha(Z)\Omega(X, Y) - \alpha(Y)\Omega(X, Z) \\ &\quad +\beta(Z)g(X, Y) - \beta(Y)g(X, Z). \end{aligned} \tag{4.25}$$

Moreover, we have

$$\begin{aligned} (\nabla_X \Omega)(Y, Z) &= X(g(JY, Z)) - g(J\nabla_X Y, Z) - g(JY, \nabla_X Z) \\ &= g(\nabla_X JY, Z) - g(J\nabla_X Y, Z) \\ &= g((\nabla_X J)Y, Z), \end{aligned} \tag{4.26}$$

which implies (4.20) holds.

■

We call the dual vector field $\alpha^\sharp$ of the Lee form α *the Lee vector field*. Let $\beta^\sharp$ be the dual vector field of β which is defined by (4.21). Then we can express (4.20) as

$$(\nabla_X J)Y = g(JX,Y)\alpha^\sharp - g(\alpha^\sharp, Y)JX + g(X,Y)\beta^\sharp - g(\beta^\sharp, Y)X. \qquad (4.27)$$

An l.c.K-manifold N is called *an l.c.K-space form* if the holomorphic sectional curvature of the section $\{X, JX\}$ at each point of N is constant. Let $N(c)$ be an l.c.K-space form with constant holomorphic sectional curvature c and let s and S be the tensor fields on $N(c)$ defined as in (4.5) and (4.6), respectively, whereby α is the Lee form of $N(c)$ and put

$$\tilde{s}(X,Y) = s(JX,Y) \qquad \text{and} \qquad g(\tilde{S}X,Y) = \tilde{s}(X,Y). \qquad (4.28)$$

Then the curvature tensor R with respect to g can be given by

$$\begin{aligned} R(X,Y)Z \;=\; & \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y \\ & +\Omega(Y,Z)JX - \Omega(X,Z)JY - 2\Omega(X,Y)JZ\} \\ & +3\{g(Y,Z)SX - g(X,Z)SY + s(Y,Z)X - s(X,Z)Y\} \\ & -\Omega(Y,Z)\tilde{S}X + \Omega(X,Z)\tilde{S}Y - \tilde{s}(Y,Z)JX + \tilde{s}(X,Z)JY \\ & +2\tilde{s}(X,Y)JZ + 2\Omega(X,Y)\tilde{S}Z. \end{aligned} \qquad (4.29)$$

4.2 CR-Submanifolds of Locally Conformal Kaehler Manifolds

In this section we investigate the integrability of the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ for the ambient manifold N is an l.c.K-manifold. We first present some closely related results which are obtained if the ambient manifold is a Kaehler one.

Lemma 4.1 [6]. *Let M be a CR-submanifold of a Kaehler manifold N. Then we have*

$$g(\nabla_X U, Y) = g(JA_{JU}X, Y) \qquad (4.30)$$

$$A_{JU}V = A_{JV}U \qquad (4.31)$$

$$A_{J\zeta}Y = -A_\zeta JY \qquad (4.32)$$

for any X tangent to M, Y in $\mathcal{D}$, U and V in $\mathcal{D}^\perp$ and ζ in ν.

Proof. Since N is a Kaehler manifold, $\tilde{\nabla}J = 0$. Thus we have

$$J\nabla_X U + Jh(X,U) = -A_{JU}X + \nabla^{\perp}_X JU \tag{4.33}$$

and so

$$-\nabla_X U - h(X,U) = -JA_{JU}X + J\nabla^{\perp}_X JU \tag{4.34}$$

for any $X \in TM$ and $U \in \mathcal{D}^{\perp}$. Since $h(X,U) \in T^{\perp}M$ and $J\nabla^{\perp}_X JU$ has no component in $\mathcal{D}$, we get (4.30).

Again by using (4.33), we obtain

$$\begin{aligned} -g(\nabla_X U, JV) - g(h(X,U), JV) &= -g(A_{JU}X, V) + g(\nabla^{\perp}_X JU, V) \\ g(A_{JV}U, X) &= g(h(X,V), JU) \\ g(A_{JV}U, X) &= g(A_{JU}V, X), \end{aligned} \tag{4.35}$$

which implies (4.31) holds.

To get (4.32), we make use of (2.25) and (3.2):

$$\begin{aligned} g(h(JY,X),\zeta) = g(\tilde{\nabla}_X JY, \zeta) &= g(J\tilde{\nabla}_X Y, \zeta) = -g(h(Y,X), J\zeta) \\ g(A_{\zeta}JY, X) &= -g(A_{J\zeta}Y, X) \\ A_{\zeta}JY &= -A_{J\zeta}Y. \end{aligned} \tag{4.36}$$

■

Lemma 4.2 [6]. *Let M be a CR-submanifold of a Kaehler manifold N. Then, for any $U, V \in \mathcal{D}^{\perp}$, we have*

$$\nabla^{\perp}_U JV - \nabla^{\perp}_V JU \in J\mathcal{D}^{\perp}. \tag{4.37}$$

Proof. For any ζ in ν and U, V in $\mathcal{D}$, we have

$$\begin{aligned} g(A_{J\zeta}V, U) &= -g(\tilde{\nabla}_V J\zeta, U) = g(\tilde{\nabla}_V \zeta, JU) \\ &= g(\nabla^{\perp}_V \zeta, JU) = -g(\zeta, \nabla^{\perp}_V JU). \end{aligned}$$

Thus we obtain

$$\begin{aligned} g(\zeta, \nabla^{\perp}_U JV - \nabla^{\perp}_V JU) &= g(A_{J\zeta}V, U) - g(A_{J\zeta}U, V) \\ &= g(h(V,U), J\zeta) - g(h(U,V), J\zeta) \\ &= 0, \end{aligned}$$

which implies $\nabla^{\perp}_U JV - \nabla^{\perp}_V JU$ has no component on ν.

■

Lemma 4.3 [6]. *The totally real distribution $\mathcal{D}^{\perp}$ of a CR-submanifold in a Kaehler manifold is integrable.*

Proof. We will show that $J[U,V] \in J\mathcal{D}^{\perp}$ for any U and V in $\mathcal{D}^{\perp}$:

$$\begin{aligned} J[U,V] &= J\tilde{\nabla}_U V - J\tilde{\nabla}_V U = \tilde{\nabla}_U JV - \tilde{\nabla}_V JU \\ &= -A_{JV}U + \nabla^{\perp}_U JV + A_{JU}V - \nabla^{\perp}_V JU. \end{aligned}$$

In virtue of (4.31) and (4.37), $J[U,V]$ is in $J\mathcal{D}^{\perp}$.

■

Lemma 4.4 [6]. *Let M be a CR-submanifold of a Kaehler manifold N. Then $\mathcal{D}$ is integrable if and only if*

$$g(h(X,JY),JU) = g(h(JX,Y),JU) \tag{4.38}$$

for any vectors X,Y in $\mathcal{D}$, and U in $\mathcal{D}^{\perp}$.

Proof. Since N is a Kaehler manifold, it has a vanishing Nijenhuis tensor field. Thus the proof follows from *Theorem 3.8* and *Proposition 3.6*.

■

Lemma 4.5 [6]. *For a CR-submanifold M in a Kaehler manifold N, the leaf $M^{\perp}$ of $\mathcal{D}^{\perp}$ is totally geodesic in N if and only if*

$$g(h(\mathcal{D},\mathcal{D}^{\perp}),J\mathcal{D}^{\perp}) = 0. \tag{4.39}$$

Proof. We have to show that $\nabla_U V \in \mathcal{D}^{\perp}$ if and only if $g(h(U,X),JV) = 0$ for any X,Y in $\mathcal{D}$ and U,V in $\mathcal{D}^{\perp}$. By using (2.25), (3.2) and (4.30), we have

$$g(h(U,X),JV) = g(A_{JV}U,X) = g(JA_{JV}U,JX) = g(\nabla_U V,JX),$$

which completes the proof.

■

Lemma 4.6 [6]. *If* (4.39) *holds and $\mathcal{D}$ is integrable, then for any X in $\mathcal{D}$ and ξ in $J\mathcal{D}^{\perp}$, we have*

$$A_{\xi}JX = -JA_{\xi}X. \tag{4.40}$$

Proof. Let U in $\mathcal{D}^\perp$ and Y in $\mathcal{D}$. Since (4.39) is satisfied, we have

$$g(h(X,U),\xi) = g(A_\xi X, U) = 0,$$

which implies $A_\xi X \in \mathcal{D}$. Also, since $\mathcal{D}$ is integrable, using (4.38), we get

$$\begin{aligned} g(h(JX,Y),\xi) &= g(h(X,JY),\xi) \\ g(A_\xi JX, Y) &= g(A_\xi X, JY) \\ g(A_\xi JX, Y) &= -g(JA_\xi X, Y), \end{aligned}$$

which completes the proof.

■

By the following Theorem, *Lemma 4.3* is generalized to CR-submanifolds in a locally conformal almost Kaehler manifold.

Theorem 4.2 [4]. *Let N be a Hermitian manifold with $d\Omega = \Omega \wedge \alpha$. Then in order that M be a CR-submanifold it is necessary that $\mathcal{D}^\perp$ be integrable.*

Proof. Let X be a vector field in $\mathcal{D}$ and U and V vector fields in $\mathcal{D}^\perp$. Then $\Omega(X,U) = 0$ and $\Omega(U,V) = 0$. Consequently, $(\Omega \wedge \alpha)(X,U,V) = 0$ and hence taking into account (3.64), we have

$$-g([U,V], JX) = 3\, d\Omega(X,U,V) = 0, \tag{4.41}$$

which implies $[U,V] \in \mathcal{D}^\perp$.

■

Now, taking into account (4.27) we prove the following Proposition:

Proposition 4.2 [12]. *In a CR-submanifold M of an l.c.K-manifold N, the distribution $\mathcal{D}^\perp$ is integrable.*

Proof. By virtue of (2.23), (2.24) and (4.27), we obtain

$$\begin{aligned} -J\nabla_X U - Jh(X,U) &= -g(\alpha^\sharp, U)JX + g(X,U)\beta^\sharp - g(\beta^\sharp, U)X \\ &\quad + A_{JU}X - \nabla^\perp_X JU \end{aligned} \tag{4.42}$$

and so,

$$\begin{aligned} \nabla_X U + h(X,U) &= g(\alpha^\sharp, U)X - g(X,U)\alpha^\sharp - g(\beta^\sharp, U)JX \\ &\quad + JA_{JU}X - J\nabla^\perp_X JU \end{aligned} \tag{4.43}$$

for any vector field X tangent to M and U in $\mathcal{D}^\perp$. Let Y be a vector field in $\mathcal{D}$. Then (4.43) takes the form

$$\begin{aligned} g(\nabla_X U, Y) &= g(\alpha^\sharp, U)g(X,Y) - g(X,U)g(\alpha^\sharp, Y) \\ &\quad -g(\beta^\sharp, U)g(JX,Y) + g(JA_{JU}X, Y). \end{aligned} \tag{4.44}$$

Now, substitute X for V, whereby V is a vector field in $\mathcal{D}^\perp$. Then (4.44) rewritten as

$$g(\nabla_V U, Y) = -g(V,U)g(\alpha^\sharp, Y) + g(JA_{JU}V, Y) \tag{4.45}$$

and hence

$$g([U,V], Y) = -g(J(A_{JU}V - A_{JV}U), Y). \tag{4.46}$$

Next, in virtue of (4.42), we have

$$\begin{aligned} g(\nabla_X U, JV) + g(h(X,U), JV) &= -g(\alpha^\sharp, U)g(JX, V) + g(X,U)g(\beta^\sharp, V) \\ &\quad -g(\beta^\sharp, U)g(X,V) + g(A_{JU}X, V) \\ &\quad -g(\nabla^\perp_X JU, V) \end{aligned} \tag{4.47}$$

and hence,

$$g(A_{JV}U, X) = g(X,U)g(\beta^\sharp, V) - g(\beta^\sharp, U)g(X,V) + g(A_{JU}V, X) \tag{4.48}$$

which yields

$$A_{JU}V - A_{JV}U = g(\beta^\sharp, U)V - g(\beta^\sharp, V)U \tag{4.49}$$

for any U and V in $\mathcal{D}^\perp$. Therefore, in virtue of (4.46) and (4.49), we conclude that $[U,V]$ belongs to $\mathcal{D}^\perp$ for any U and V in $\mathcal{D}^\perp$.

■

For any vector fields X tangent to M, Y in $\mathcal{D}$ and ζ in ν, by using (4.27), we obtain

$$\begin{aligned} g(\nabla_X Y, J\zeta) + g(h(X,Y), J\zeta) &= g(\alpha^\sharp, \zeta)g(JX,Y) - g(\alpha^\sharp, Y)g(JX, \zeta) \\ &\quad +g(\beta^\sharp, \zeta)g(X,Y) - g(X,\zeta)g(\beta^\sharp, Y) \\ &\quad -g(\nabla_X JY, \zeta) - g(h(X,JY), \zeta), \end{aligned} \tag{4.50}$$

that is,

$$
\begin{aligned}
g(A_{J\zeta}Y, X) + g(A_{\zeta}JY, X) &= -g(\alpha^{\sharp}, \zeta)g(JY, X) + g(\beta^{\sharp}, \zeta)g(Y, X) \\
A_{\zeta}JY + A_{J\zeta}Y &= g(\beta^{\sharp}, \zeta)Y - g(\alpha^{\sharp}, \zeta)JY \qquad (4.51)
\end{aligned}
$$

which implies $A_{\zeta}JY + A_{J\zeta}Y$ belongs to $\mathcal{D}$. Notice that, by virtue of *Lemma 4.1*, this sum is identically zero in the case N is a Kaehler manifold.

Proposition 4.3 [12]. *The distribution $\mathcal{D}$ of a submanifold M of an l.c.K-manifold N is integrable if and only if*

$$
g(h(X, JY) - h(Y, JX) - 2g(JX, Y)\alpha^{\sharp}, JU) = 0 \qquad (4.52)
$$

for any X and Y in $\mathcal{D}$ and U in $\mathcal{D}^{\perp}$.

Proof. From (4.27), we obtain

$$
\begin{aligned}
\tilde{\nabla}_X JY - J\tilde{\nabla}_X Y &= g(JX, Y)\alpha^{\sharp} - g(\alpha^{\sharp}, Y)JX \\
&\quad -g(X, Y)\beta^{\sharp} - g(\beta^{\sharp}, Y)X, \qquad (4.53) \\
\tilde{\nabla}_Y JX - J\tilde{\nabla}_Y X &= g(JY, X)\alpha^{\sharp} - g(\alpha^{\sharp}, X)JY \\
&\quad -g(X, Y)\beta^{\sharp} - g(\beta^{\sharp}, X)Y, \qquad (4.54)
\end{aligned}
$$

whereby X and Y are vector fields in $\mathcal{D}$. Next, by subtracting (4.54) from (4.53), we get

$$
\begin{aligned}
h(X, JY) - h(Y, JX) - 2g(JX, Y)\alpha^{\sharp} &= J[X, Y] - \nabla_X JY + \nabla_Y JX \\
&\quad -g(\alpha^{\sharp}, Y)JX + g(\alpha^{\sharp}, X)JY \\
&\quad -g(\beta^{\sharp}, Y)X + g(\beta^{\sharp}, X)Y, \quad (4.55)
\end{aligned}
$$

which implies if $\mathcal{D}$ is integrable (4.52) holds and vice versa.

■

Proposition 4.4 [12]. *The leaf $M^{\perp}$ of the distribution $\mathcal{D}^{\perp}$ of a CR-submanifold M of an l.c.K-manifold N is totally geodesic in M if and only if*

$$
g(A_{JV}U + g(U, V)\beta^{\sharp}, X) = 0 \qquad (4.56)
$$

for any X in $\mathcal{D}$ and U and V in $\mathcal{D}^{\perp}$.

Proof. We have to show that $\nabla_U V \in \mathcal{D}^\perp$ for any U and V in $\mathcal{D}^\perp$. From (4.45), we get

$$\begin{aligned} g(\nabla_U V, JX) &= g(JA_{JV}U + g(U,V)J\beta^\sharp, JX) \\ &= g(A_{JV}U + g(U,V)\beta^\sharp, X), \end{aligned} \tag{4.57}$$

whereby X is a vector field in $\mathcal{D}$.

■

Proposition 4.5 [12]. *If in a CR-submanifold M of an l.c.K-manifold N, the distribution $\mathcal{D}$ is integrable and the leaf $M^\perp$ of $\mathcal{D}^\perp$ is totally geodesic in M, then we have*

$$A_{JU}JX + JA_{JU}X - 2g(\alpha^\sharp, JU)JX + g(\alpha^\sharp, X)U + g(\beta^\sharp, X)JU = 0 \tag{4.58}$$

for any X in $\mathcal{D}$ and U in $\mathcal{D}^\perp$.

Proof. To prove (4.58), it is sufficient to show that

$$\begin{aligned} &g(A_{JU}JX + JA_{JU}X - 2g(\alpha^\sharp, JU)JX \\ &\qquad +g(\alpha^\sharp, X)U + g(\beta^\sharp, X)JU, \mathcal{D} \oplus \mathcal{D}^\perp \oplus J\mathcal{D}^\perp) = 0. \end{aligned} \tag{4.59}$$

For any vector field Y in $\mathcal{D}$, in virtue of (4.52), we have

$$\begin{aligned} g(A_{JU}JX, Y) + g(JA_{JU}X, Y) & \\ -2g(\alpha^\sharp, JU)g(JX, Y) + g(\alpha^\sharp, X)g(U, Y) & \\ +g(\beta^\sharp, X)g(JU, Y) &= g(h(JX, Y), JU) \\ &\quad -g(h(X, JY), JU) \\ &\quad -2g(\alpha^\sharp, JU)g(JX, Y) \\ &= 0. \end{aligned}$$

Next, for any V in $\mathcal{D}^\perp$, in virtue of (4.56), we have

$$\begin{aligned} g(A_{JU}JX, V) + g(JA_{JU}X, V) & \\ -2g(\alpha^\sharp, JU)g(JX, V) + g(\alpha^\sharp, X)g(U, V) & \\ +g(\beta^\sharp, X)g(JU, V) &= g(A_{JU}JX, V) + g(\beta^\sharp, JX)g(U, V) \\ &= 0. \end{aligned}$$

Now, for any V in $\mathcal{D}^{\perp}$, in virtue of (4.56), we have

$$\begin{aligned} g(A_{JU}JX, JV) + g(A_{JU}X, V) & \\ -2g(\alpha^{\sharp}, JU)g(X, V) + g(\alpha^{\sharp}, X)g(U, JV) & \\ +g(\beta^{\sharp}, X)g(U, V) &= g(A_{JU}X, V) + g(\beta^{\sharp}, X)g(U, V) \\ &= 0. \end{aligned}$$

■

5. CONCLUSION AND RECOMMENDATIONS

In 1978, the concept of *CR-submanifold* of an almost Hermitian manifold was introduced by Aurel Bejancu as follows:

Let N be an almost Hermitian manifold and let J be the almost complex structure of N. A real submanifold M of N is called *a CR-submanifold* if there exists a differentiable distribution $\mathcal{D}$ on M satisfying $J(\mathcal{D}_x) = \mathcal{D}_x$ and $J(\mathcal{D}_x^\perp) \subset T_x^\perp M$ for each $x \in M$, whereby $\mathcal{D}^\perp$ is the complementary orthogonal distribution to $\mathcal{D}$ and $T_x^\perp M$ is the normal space to M at x.

After that, the definition was soon extended to other ambient spaces. The definition of locally conformal Kaehler manifolds is given as follows:

Let N be a real $2n$-dimensional Hermitian manifold with the structure (J, g), whereby J is an almost complex structure and g is a Hermitian metric. Then N is *a locally conformal Kaehler manifold (an l.c.K-manifold)* if there is an open cover $\{U_i\}_{i\in I}$ of N and a family $\{\sigma_i\}_{i\in I}$ of C^∞ functions $\sigma_i \colon U_i \to \mathbf{R}$ so that each local metric $g_i = e^{-2\sigma_i} g|_{U_i}$ is a Kaehler metric.

In this thesis, we present the results on the integrability of the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ which are obtained if the ambient space N is a locally conformal Kaehler one.

Let $M_\top$ (resp. $M_\perp$) be a holomorphic (resp. totally real) submanifold in an l.c.K-manifold N. We consider a warped product submanifold of the form $M_1 = M_\perp \times_f M_\top$ with a warping function $f(> 0) \in C^\infty(M_\perp)$, whereby $C^\infty(M_\perp)$ denotes the set of all differentiable functions on $M_\perp$. We call such a submanifold *a warped product CR-submanifold* in an l.c.K-manifold N.

In future, we aim to study the warped product CR-submanifolds in locally conformal Kaehler manifolds.

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